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On the approximability and the selection of particle shape functions

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Summary. Particle methods, also known as meshless or meshfree methods, have become popular in approximating solutions of partial differential equations, especially in the engineering community. These methods do not employ a mesh, or use it minimally, in the construction of shape functions. There is a wide variety of classes of shape functions that can be used in particle methods. In this paper, we primarily address the issue of selecting a class of shape functions, among this wide variety, that would yield efficient approximation of the unknown solution. We have also made several comments and observations on the order of convergence of the interpolation error, when these shape functions are used; specifically, we have shown that the interpolation error estimate, for certain classes of shape functions, may not indicate the actual order of convergence of the approximation error.

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1 Introduction

Recently, a class of methods known as particle methods, has attracted attention in the context of approximating solutions of partial differential equations that arise in computational mechanics. These methods are also referred to as

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meshless methods or meshfree methods. The main feature of these methods is that they do not employ a mesh, or use a mesh only minimally, in the construction of shape functions. Particle methods have potential for efficiently handling certain difficult problems, *e.g.,* problems with large deformations, or crack propagation [11, 15].

Several particle methods have been developed over the last decade, *e.g.,* Moving Least Square particle methods (MLS), Reproducing Kernel Particle methods (RKP), Moving Least Square Kernel particle methods (MLSK). For an overview of these methods, we refer to [3, 7, 14, 16–18]. Particle methods are related to Generalized Finite Element Method [9, 19, 25]. One of the main differences between various particle methods is that they use different shape functions. Moreover, for any particular particle method, say the RKP method, a wide variety of shape functions can be constructed by using various weight or window functions. It is important to address the issue of selecting a class of shape functions, among this wide variety of possible shape functions, that would yield an efficient approximation of the solution of a particular problem, or a class of problems.

In this paper, we have assumed that particles are uniformly distributed, and have considered shape functions that are translation invariant. These shape functions were first introduced and analyzed in [4, 23, 24]. Translation invariant shape functions can also be constructed by following, with some adjustments, the procedures for constructing shape functions used in the papers mentioned in the previous paragraph. This will be further discussed in the next section. Some shape functions, *e.g.,* RKP and MLS shape functions, can be defined for non-uniformly as well as uniformly distributed particles, but they are, in general, not translation invariant. We will refer to such shape functions as standard particle shape functions, *e.g., standard* RKP shape functions. We note, however, that for uniformly distributed particles, the shape functions, *e.g.*, RKP shape functions, are translation invariant in the interior of the domain, sufficiently away from the boundary, of the underlying problem. But, they are translation invariant over the entire domain when the domain is $\mathbb{R}^n$. Though, in this paper, we primarily discuss the translation invariant shape functions, we will often make comments on *standard* RKP shape functions.

We have presented the following results and observations in this paper:

- A procedure for choosing a class of translation invariant particle shape functions, among a given collection of such classes, has been proposed. This choice of shape functions will yield the smallest value for the usual Sobolev norm of the "interpolation" error, when the interpolated function is included in a Sobolev space of high order (*i.e.,* when the function is smooth). This information is relevant for the selection of shape functions; it was shown in [5, 6] that the selection of efficient shape functions depends strongly on the function space inclusions of the approximated function. This selection procedure is primarily based on Theorem 3.1.

- In certain situations, interpolation error estimates do not indicate the correct order of convergence of the approximation error, and is pessimistic.
- For translation invariant shape functions, the interpolation error (for smooth functions) may decrease at a higher rate, in the pre-asymptotic range, than is predicted by the theory.
- The performances of translation invariant shape functions and standard RKP shape functions have been compared in the context of interpolation and projection. It was observed that standard RKP shape functions may yield erratic error behavior near the boundary.

We did not consider the imposition of Dirichlet boundary conditions. Hence our analysis is relevant to boundary value problems on $\mathbb{R}^n$, or on a bounded domain with Neumann boundary conditions. We note that imposition of the Dirichlet boundary conditions in particle methods is not easy in higher dimensions [13]; some work has been done in this area (see [7],[22] and the references in [26]). Also, we did not consider implementational aspects of particle methods.

Some of the results in Section 3 of this paper will also appear in [8]. In this paper, we have not included proofs of the results that appeared in the complete form in [8].

We now present some notions and notations that will be used throughout this paper. We will consider multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, which are ordered collections of non-negative integers, α_i. For any $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, multi-index α, and a function u, we write

$$|\alpha| = \sum_{i=1}^{n} \alpha_i, \quad \alpha! = \alpha_1!\alpha_2! \cdots \alpha_n!, \quad x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n},$$

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}.$$

For a domain $\Omega \subset \mathbb{R}^n$, we denote the usual Sobolev space by $H^k(\Omega)$; for $u \in H^k(\Omega)$, the norm is

$$\|u\|_{k,\Omega}^2 = \sum_{|\alpha| \le k} \int_\Omega |D^\alpha u|^2 \, dx,$$

and the semi-norm is

$$|u|_{k,\Omega}^2 = \sum_{|\alpha| = k} \int_\Omega |D^\alpha u|^2 \, dx.$$

We also use, for a vector valued function $\vec{u}$,

$$\|\vec{u}\|_{0,\Omega}^2 = \int_\Omega \vec{u} \cdot \vec{u}\, dx.$$

We briefly describe the organization of this paper. In Section 2, we discuss translation invariant shape functions, together with an example. In Section 3, we discuss interpolation by these shape functions in a bounded domain, and present the main theoretical result (Theorem 3.1). In Section 4, we propose a procedure for choosing efficient shape functions, and numerically test its validity in the case of interpolation. In Section 5, we numerically compare the approximation error with interpolation error, and also show that the interpolation error may decrease at a higher rate (than predicted) in the pre-asymptotic range. In Section 6, we compare the performance of translation invariant shape functions with standard RKP shape functions.

2 Particle shape functions

Let

$$\mathbb{Z}^n \equiv \{j = (j_1, j_2, \ldots, j_n) : j_1, \ldots, j_n \text{ integers}\}$$

be the integer lattice, and define, for $0 < h \leq 1$,

$$x_j^h \equiv \{(j_1 h, \ldots, j_n h) = hj, \text{ where } j = (j_1, \ldots, j_n) \in \mathbb{Z}^n\}.$$

We refer to the x_j^h's as *uniformly distributed particles* in $\mathbb{R}^n$. We often construct shape functions as follows: Let $\phi(x) \in H^q(\mathbb{R}^n)$, where $q > \frac{n}{2}$ and $q \geq 1$, be a function with compact support and suppose

$$\eta \equiv \operatorname{supp} \phi(x) \subset B_R(0) = \{x : \|x\| < R\},$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$. We note that $\phi(x)$ is a continuous function. We assume that $0 \in \mathring{\eta}$ (interior of η). The function ϕ is called the *basic shape function*. Then, for $j \in \mathbb{Z}^n$, we define

$$\phi_j^h(x) = \phi\left(\frac{x}{h} - j\right). \tag{2.1}$$

It is immediate that

$$\eta_j^h \equiv \operatorname{supp} \phi_j^h(x) \subset B_{Rh}(x_j^h) = \{x : \|x - x_j^h\| < Rh\}.$$

Note that $x_j^h \in \mathring{\eta}_j^h \cdot \phi_j^h$ is called the *particle shape function* associated with the particle x_j^h (and the basic shape function ϕ). We note, however, that there are other ways to construct particle shape functions.

We first observe that the uniformly distributed particles satisfy $x^h_{j+i} = x^h_j + x^h_i$, and the associated particle shape functions are translation invariant in the sense that

$$\phi^h_{j+i}(x) = \phi^h_j(x - x^h_i);$$

in the sequel, these shape functions will be referred to as translation invariant particle shape function. Moreover, from a standard scaling argument, we have

$$\|\phi^h_j\|_{1,\mathbb{R}^n} \leq h^{n/2-1}\|\phi\|_{1,\mathbb{R}^n}. \tag{2.2}$$

The basic shape function $\phi(x)$ is called *Quasi-Reproducing of order k* if for any multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, with $|\alpha| \leq k$,

$$\sum_{j\in\mathbb{Z}^n} j^\alpha \phi(x - j) = \lambda\, x^\alpha + q^\alpha(x), \quad \text{for all } x \in \mathbb{R}^n, \tag{2.3}$$

where $\lambda \neq 0$ and $q^\alpha(x)$ is a polynomial with degree $< |\alpha|$. If (2.3) holds with $\lambda = 1$ and $q^\alpha(x) = 0$, then the basic shape function $\phi(x)$ is called *Reproducing of order k*.

If $\phi(x)$ is quasi-reproducing of order k (respectively reproducing of order k), then equivalently, the corresponding particle shape functions $\{\phi^h_j\}$ are also quasi-reproducing of order k (respectively reproducing of order k), *i.e.*, for $|\alpha| \leq k$,

$$\sum_{j\in\mathbb{Z}^n} (x^h_j)^\alpha \phi^h_j(x) = \lambda\, x^\alpha + \bar{q}^\alpha_h(x), \quad \text{for all } x \in \mathbb{R}^n, \tag{2.4}$$

where $\lambda \neq 0$ and $\bar{q}^\alpha_h(x) = h^{|\alpha|} q^\alpha(x/h)$ (respectively, (2.4) holds with $\lambda = 1$ and $\bar{q}^\alpha_h(x) = 0$). We note that $\{\phi^h_j\}$ are reproducing of order k if and only if

$$\sum_{j\in\mathbb{Z}^n} p(x^h_j)\phi^h_j(x) = p(x), \ \text{ for all } p \in \mathcal{P}^k(\mathbb{R}^n), \tag{2.5}$$

where $\mathcal{P}^k(\mathbb{R}^n)$ is the space of polynomials of degree $\leq k$. We remark that in assuming (2.3) (or (2.4)), we have implicitly assumed that $\cup_{j\in\mathbb{Z}^n} \mathring{\eta}^h_j = \mathbb{R}^n$.

The basic shape function $\phi(x)$ (and consequently the corresponding particle shape functions $\phi^h_j(x)$) is called *Strongly r-Reproducing of order k*, for $1 \leq r \leq k+1$, if $\phi(x)$ is reproducing of order k, and

$$\begin{aligned} &\text{(a)} \ \ q(x) \neq \sum_{j\in\mathbb{Z}^n} q(j)\phi(x - j) \\ &\text{(b)} \ \ q(x) - \sum_{j\in\mathbb{Z}^n} q(j)\phi(x - j) \notin \mathcal{P}^{r-1}(\mathbb{R}^n) \end{aligned} \tag{2.6}$$

for any polynomial $q(x)$ of degree $(k+1)$. For $r = 0$, the definition of strongly r-reproducing shape functions of order k does not include item (b) in (2.6). We note that in 1-d, if the basic shape function $\phi(x)$ is reproducing of order k and not quasi-reproducing of any higher order, then $\phi(x)$ is strongly r-reproducing of order k, for all $0 \leq r \leq k+1$.

Examples of particle shape functions reproducing of order k

Various types of particle shape functions are used in practice; *e.g.,* RKP shape functions and MLS particle shape functions, as mentioned in Section 1. We briefly describe RKP shape functions. We first describe translation invariant RKP shape functions.

Let $\omega(x) \geq 0$ be a continuous function with

$$\eta \equiv \operatorname{supp} \omega(x) = \overline{B}_R(0).$$

The function $\omega(x)$ is called a weight function (or window function). For each particle x_j^h, we associate the weight function $\omega_j^h(x)$ defined by

$$\omega_j^h(x) = \omega\left(\frac{x - x_j^h}{h}\right) = \omega\left(\frac{x}{h} - j\right).$$

For a given positive integer k, the RKP shape function $\phi_j^h(x)$, associated with the particle x_j^h, is defined by

$$\phi_j^h(x) = \omega_j^h(x) \sum_{|\alpha| \leq k} (x - x_j^h)^\alpha b_\alpha^h(x), \tag{2.7}$$

where the $b_\alpha^h(x)$ are chosen so that

$$\sum_{j \in \mathbb{Z}^n} p(x_j^h)\phi_j^h(x) = p(x), \text{ for all } p \in \mathcal{P}^k(\mathbb{R}^n). \tag{2.8}$$

This gives rise to a linear system in $b_\alpha^h(x)$, namely

$$\sum_{|\alpha| \leq k} m_{\alpha+\beta}^h(x) b_\alpha^h(x) = \delta_{|\beta|,0}, \text{ for } |\beta| \leq k, \tag{2.9}$$

where

$$m_\alpha^h(x) = \sum_{j \in \mathbb{Z}^n} \omega_j^h(x)(x - x_j^h)^\alpha, \tag{2.10}$$

and $\delta_{|\beta|,0}$ is the Kronecker delta. For sufficient conditions to ensure unique solvability of (2.9), see [13].

If we let $\phi(x) = \phi_0^h(x)$ with $h = 1$, then $\phi_j^h(x)$ satisfies (2.1), *i.e.,* $\phi_j^h(x) = \phi(\frac{x}{h} - j)$ (see [8]). Thus RKP shape functions $\phi_j^h(x)$ (cf. (2.7)), are particle shape functions associated with particles x_j^h, and they are translation invariant and reproducing of order k. We will refer to them as *translation invariant RKP shape functions*.

The construction of RKP shape functions as given in [13], for uniformly distributed particles, is slightly different than described above in (2.7)–(2.10), and is usually done in the context of a bounded domain $\Omega \subset \mathbb{R}^n$. These shape functions $\phi_j^h(x)$ are defined as in (2.7), but only for $j \in A_\Omega$, where

$$A_\Omega = \{j \in \mathbb{Z}^n : x_j^h \in \bar{\Omega}\},$$

and (2.8) and (2.10) are replaced by

$$\sum_{j \in A_\Omega} p(x_j^h)\phi_j^h(x) = p(x), \text{ for all } p \in \mathcal{P}^k(\Omega) \tag{2.11}$$

and

$$m_\alpha^h(x) = \sum_{j \in A_\Omega} \omega_j^h(x)(x - x_j^h)^\alpha, \tag{2.12}$$

respectively. By $\mathcal{P}^k(\Omega)$ we mean polynomials of degree $\le k$ restricted to Ω, and hence (2.11) is required to hold only for $x \in \Omega$. Thus, only the particles that are contained in $\bar{\Omega}$ are used in the construction of $\phi_j^h(x)$. Note that these shape functions only "reproduce" polynomials of degree $\le k$ in Ω, and they are *not translation invariant*. We will refer to these shape functions as *standard* RKP shape functions throughout this paper. We remark that for uniformly distributed particles, the standard RKP shape functions (constructed with respect to Ω) corresponding to the particles in $\Omega_0 \subset \Omega$, where Ω_0 is sufficiently away from $\partial\Omega$, are translation invariant with respect to the particles in Ω_0. We further remark that standard RKP shape functions are same as translation invariant RKP shape functions when $\Omega = \mathbb{R}^n$.

We note that the RKP shape functions, translation invariant or standard, depend on the weight function $\omega(x)$. Other particle shape functions, which we have not described here, also depend on such weight functions. The commonly used weight functions in 1-d are:

(a) Gaussian weight function:

$$\omega(x) = \begin{cases} \dfrac{e^{\delta(x/R)^2} - e^\delta}{1 - e^\delta}, & |x| \le R, \\ 0, & |x| \ge R, \end{cases} \tag{2.13}$$

where $\delta > 0$;

(b) Cubic spline weight function:

(2.14)
$$\omega(x) = \begin{cases} \frac{2}{3} - 4(x/R)^2 + 4(|x|/R)^3, & |x| \le R/2, \\ \frac{4}{3} - 4(|x|/R) + 4(x/R)^2 - \frac{4}{3}(|x|/R)^3, & R/2 \le |x| \le R, \\ 0, & |x| > R; \end{cases}$$

(c) Conical weight function:

$$\omega(x) = \begin{cases} [1 - (x/R)^2]^l, & |x| \le R, \\ 0, & |x| > R, \end{cases} \tag{2.15}$$

where $l = 1, 2 \ldots$.
We note that one may consider non-symmetric versions of some of these weight functions, as was done in [2].

In higher dimension, the weight function $\omega(x)$ can be constructed from the one dimensional weight function $\omega(x)$ as $\omega(x) = \omega(\|x\|)$, where $\|x\|$ is the Euclidean length of x, or as $\omega(x) = \prod_{i=1}^n \omega(x_i)$, where $x = (x_1, x_2, \ldots, x_n)$.

We now introduce some notation that will be used in this paper. Let I_j^h be a *cell* centered at the particle x_j^h, given by

$$I_j^h = \{x : \|x - x_j^h\|_\infty \equiv \max_{i=1,\ldots,n} |x_i - x_{j_i}^h| \le h/2\},$$

where $x_{j_i}^h$ is the i^{th} coordinate of x_j^h. For each I_j^h, we define

$$A_j^h = \{m \in \mathbb{Z}^n : \overset{\circ}{\eta}{}_m^h \cap I_j^h \neq \emptyset\}$$

and

$$B_j^h = \{\cup_{m \in A_j^h} B_{Rh}(x_m^h)\} \cup I_j^h.$$

The cardinality of A_j^h is finite, and is bounded independent of j and h. Also there exists $\bar{R} > 0$, independent of j and h, such that $B_j^h \subset \tilde{B}_j^h \equiv B_{\bar{R}h}(x_j^h)$.

3 Interpolation by translation invariant particle shape functions in a bounded domain in $\mathbb{R}^n$

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz continuous boundary. Let $\{x_j^h\}$ be the set of uniformly distributed particles in $\mathbb{R}^n$, and consider the associated translation invariant particle shape functions $\{\phi_j^h\}$, which we assume to be reproducing of order k. In this section, we will consider a smooth function $u(x)$ defined in Ω and study the interpolation error between u and $\tilde{\mathcal{I}}_h u$, where $\tilde{\mathcal{I}}_h u$ is the “interpolant” of u in terms of ϕ_j^h.

We first define the interpolant $\tilde{\mathcal{I}}_h u$, when u is defined on $\mathbb{R}^n$, as follows:

$$(\tilde{\mathcal{I}}_h u)(x) = \sum_{j \in \mathbb{Z}^n} u(x_j^h)\phi_j^h(x).$$

It is immediate that this definition can be stated as

$$(\tilde{\mathcal{I}}_h u)(x) = \sum_{j \in A_x^h} u(x_j^h)\phi_j^h(x), \tag{3.1}$$

where

$$A_x^h = \{j \in \mathbb{Z}^n : x \in \mathring{\eta}_j^h\}$$

is the *influence set* of x. If $p \in \mathcal{P}^k(\mathbb{R}^n)$, then $p(x_j^h)$ is defined for all j, and from (2.5) we have,

$$\sum_{j \in A_x^h} p(x_j^h)\phi_j^h(x) = \sum_{j \in \mathbb{Z}^n} p(x_j^h)\phi_j^h(x) = p(x), \text{ for all } x \in \mathbb{R}^n,$$

i.e., $\tilde{\mathcal{I}}_h p = p$. In (3.1) we only need that $u(x_j^h)$ is defined for all $j \in A_x^h$.

Now, let $u \in H^s(\Omega)$, with $s > n/2$. For some $x \in \Omega$, the particles x_j^h for $j \in A_x^h$ may be outside Ω, and $u(x_j^h)$ is not defined. To define $\tilde{\mathcal{I}}_h u(x)$ in this situation, we need an extension $\bar{u}$ of u to an open ball B_{R_0}, with R_0 sufficiently large, satisfying $\bar{u} \in H^s(B_{R_0})$. Then we define

$$(\tilde{\mathcal{I}}_h u)(x) \equiv (\tilde{\mathcal{I}}_h \bar{u})(x) = \sum_{j \in A_x^h} \bar{u}(x_j^h)\phi_j^h(x), \text{ for all } x \in B_{R_0 - Rh}, \tag{3.2}$$

where we assume that $\Omega \subset B_{R_0-Rh}$. For an extension $\bar{u}$, when $u \in H^s(\Omega)\backslash \mathcal{P}^{k+1}(\Omega)$, we may use $\bar{u} = Eu$, where $E : L_2(\Omega) \to L_2(B_{R_0})$ is an extension operator satisfying $Eu|_\Omega = u$ for all $u \in L_2(\Omega)$, such that, if $u \in H^m(\Omega)$, then $Eu \in H^m(B_{R_0})$ and

$$\|\bar{u}\|_{H^m(B_{R_0})} = \|Eu\|_{H^m(B_{R_0})} \leq C_m \|u\|_{H^m(\Omega)}, \; m = 0, 1, \cdots . \tag{3.3}$$

The existence of such an operator is well known (see [21]). When $u \in \mathcal{P}^{k+1}(\Omega)$, we use $\bar{u} = u$, *i.e.*, $\bar{u}$ is its own extension. Because of finite dimensionality of polynomials of degree $k+1$, (3.3), with Eu replaced by $\bar{u}$, is also satisfied in this situation. Thus $(\tilde{\mathcal{I}}_h u)(x)$, for $x \in \Omega$, will depend on several values $\bar{u}(x_j^h)$, where the particle x_j^h is just outside Ω. We note that $\tilde{\mathcal{I}}_h u$ is not an interpolant of u in the usual sense, since, generally, $\phi_i^h(x_j^h) \neq \delta_{ij}$, and therefore $(\tilde{\mathcal{I}}_h u)(x_j^h) \neq u(x_j^h)$. We further note that $\tilde{\mathcal{I}}_h u$ depends on the extension of u, and since there are many possible extensions, $\tilde{\mathcal{I}}_h u$ is not unique. But once a particular extension $\bar{u}$ of u is chosen, $\tilde{\mathcal{I}}_h u$ is unique. $\tilde{\mathcal{I}}_h$ is not linear in u but is linear in $\bar{u}$.

In this section, we will investigate the interpolation error $\|u - \tilde{\mathcal{I}}_h u\|_{1,\Omega}$. Towards this end, we define the function

$$
\begin{aligned}
\xi_\alpha^h(x) &\equiv x^\alpha - \tilde{\mathcal{I}}_h(x^\alpha) \\
&= x^\alpha - \sum_{i\in\mathbb{Z}^n} (x_i^h)^\alpha \phi_i^h(x), \quad |\alpha| = k+1;
\end{aligned} \tag{3.4}
$$

we will also use

$$
\xi_\alpha(x) \equiv \xi_\alpha^1(x) = x^\alpha - \sum_{i\in\mathbb{Z}^n} i^\alpha \phi(x-i), \quad |\alpha| = k+1.
$$

In 1-d, we will write these functions as $\xi_{k+1}^h(x)$ and $\xi_{k+1}(x)$, respectively.

We begin with certain results about these functions.

Lemma 3.1 *$\xi_\alpha^h(x)$, with $|\alpha| = k+1$, is periodic, i.e.,*

$$
\xi_\alpha^h(x + x_j^h) = \xi_\alpha^h(x), \quad \textit{for any } x_j^h \textit{ for all } x \in \mathbb{R}^n. \tag{3.5}
$$

The proof of this result was given in [8]; we do not repeat it here. We note, however, that this result plays a central role in the analysis presented in this section.

We note that using $h = 1$ in Lemma 3.1, we have

$$
\xi_\alpha(x + j) = \xi_\alpha(x), \quad \text{for any } j \in \mathbb{Z}^n.
$$

Lemma 3.2 *Let $\alpha = \alpha(i)$, $i = 1, \cdots, M_k$, be an enumeration of the multi-index α with $|\alpha(i)| = k+1$. Let I_j^h be the cell centered at the particle x_j^h. Then, for $d_\alpha \in \mathbb{R}$, we have*

$$
\left\| \sum_{|\alpha|=k+1} \frac{1}{\alpha!} d_\alpha \xi_\alpha^h(x) \right\|_{1,I_j^h}^2 = h^{2k+n} \mathbb{V}^T (A + h^2 B) \mathbb{V}, \tag{3.6}
$$

where $\mathbb{V} = [d_{\alpha(1)}, d_{\alpha(2)}, \ldots, d_{\alpha(M_k)}]^T$ and A, B are $M_k \times M_k$ matrices given by

$$
A_{lm} = \int_I \frac{1}{\alpha(l)!\alpha(m)!} \nabla\xi_{\alpha(l)} \cdot \nabla\xi_{\alpha(m)}\, dx, \tag{3.7}
$$

$$
B_{lm} = \int_I \frac{1}{\alpha(l)!\alpha(m)!} \xi_{\alpha(l)} \xi_{\alpha(m)}\, dx, \tag{3.8}
$$

respectively, and $I = [-1/2, 1/2]^n$.

The proof of this result was given in [8]; we do not repeat it here.

Remark 3.1 In 1-d, (3.6) reduces to

$$
\|\xi_{k+1}^h\|_{1,I_j^h}^2 = h^{2k+1} \left[\int_{-1/2}^{1/2} (\xi_{k+1}')^2\, dy + h^2 \int_{-1/2}^{1/2} \xi_{k+1}^2\, dy \right]. \tag{3.9}
$$

Remark 3.2 We note that if $\xi_\alpha(x) = C_\alpha$, where C_α is a constant (for $|\alpha| = k+1$), then the matrix A is the zero matrix. Also the matrix B is zero matrix if $\xi_\alpha(x) = 0$ (for $|\alpha| = k+1$), *i.e.,* if the shape functions are reproducing of order $(k+1)$.

Remark 3.3 We note that the matrix A is positive semi-definite, and its smallest eigenvalues may be 0. But if the basic shape function $\phi(x)$ is *strongly r-reproducing* of order k with $r = 1$, then A is positive-definite. To briefly see this, let $\mathbb{V} = \{d_\alpha\}$ be a non-zero vector in $\mathbb{R}^{M_k}$. Then

$$\mathbb{V}^T A \mathbb{V} = \left\| \sum_{|\alpha|=k+1} \frac{1}{\alpha!} d_\alpha \nabla \xi_\alpha \right\|_{0,I}^2, \tag{3.10}$$

where $I = [-1/2, 1/2]^n$. Suppose $\mathbb{V}^T A \mathbb{V} = 0$. Then

$$\sum_{|\alpha|=k+1} \frac{1}{\alpha!} d_\alpha \xi_\alpha = q(x) - \sum_{j\in\mathbb{Z}^n} q(j)\phi(x-j) = C,$$

where $q(x) = \sum_{|\alpha|=k+1} \frac{1}{\alpha!} d_\alpha x^\alpha$ is a polynomial of degree $(k+1)$ and C is a constant. This is a contradiction, since it violates (2.6). Therefore $\mathbb{V}^T A \mathbb{V} \neq 0$, which implies that A is positive definite.

Remark 3.4 The matrix B is also positive semi-definite. But if the basic shape function $\phi(x)$ is strongly r-reproducing of order k with $r = 0$, then we can also show, as in Remark 3.3, that B is positive definite.

Lemma 3.3 *Let I_j^h be the cell centered at the particle x_j^h, and consider the corresponding set $\tilde{B}_j^h$. Suppose $u \in H^{k+2+q}(\tilde{B}_j^h)$, with $q > \frac{n}{2}$, $n \geq 2$. Then,*
(a) for any $\delta > 0$,

$$\begin{aligned} \|u - \tilde{\mathcal{I}}_h u\|_{1,I_j^h}^2 &\leq (1+\delta^2) \left\| \sum_{|\alpha|=k+1} \frac{1}{\alpha!} (D^\alpha u)(x_j^h) \xi_\alpha^h(x) \right\|_{1,I_j^h}^2 \\ &\quad + \left(1 + \frac{1}{\delta^2}\right) C h^{2k+2} \sum_{|\alpha|=k+2} \|D^\alpha u\|_{q,\tilde{B}_j^h}^2; \end{aligned} \tag{3.11}$$

(b) for any $\delta > 0$,

$$\begin{aligned} &\left\| \sum_{|\alpha|=k+1} \frac{1}{\alpha!} (D^\alpha u)(x_j^h) \xi_\alpha^h(x) \right\|_{1,I_j^h}^2 \\ &\quad \leq (1+\delta^2) \|u - \tilde{\mathcal{I}}_h u\|_{1,I_j^h}^2 + \left(1 + \frac{1}{\delta^2}\right) C h^{2k+2} \sum_{|\alpha|=k+2} \|D^\alpha u\|_{q,\tilde{B}_j^h}^2. \end{aligned} \tag{3.12}$$

Note: The particles x_j^h considered in this lemma are such that $\tilde{B}_j^h \subset B_{R_0}$. Also, the function u should be thought of as the extended function $\bar{u}$ to B_{R_0}.

Proof. (a) Let $x \in I_j^h$. Since $u \in H^{k+2+q}(\tilde{B}_j^h)$, with $q > \frac{n}{2}$, from Sobolev's inequality ($|v(x)| \le C\|v\|_{q,\tilde{B}_j^h}$) we know that $D^\alpha u(x)$, for $|\alpha| \le k+2$, has point values in $\tilde{B}_j^h$. Therefore from Taylor's theorem, we have

$$\begin{aligned} u(x) = & \sum_{|\alpha| \le k} \frac{1}{\alpha!}(D^\alpha u)(x_j^h)(x - x_j^h)^\alpha \\ & + \sum_{|\alpha|=k+1} \frac{1}{\alpha!}(D^\alpha u)(x_j^h)(x - x_j^h)^\alpha + R_{k+1}u(x), \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} R_{k+1}u(x) = (k+2) & \sum_{|\alpha|=k+2} (x - x_j^h)^\alpha \\ & \times \int_0^1 \frac{s^{k+1}}{\alpha!}(D^\alpha u)(x + s(x_j^h - x))\, ds. \end{aligned} \tag{3.14}$$

We first note that, using the translation invariance of $\{\phi_i^h\}$,

$$\tilde{\mathcal{I}}_h(x - x_j^h)^\alpha = \sum_{i \in \mathbb{Z}^n} (x_{i-j}^h)^\alpha \phi_i^h(x) = \sum_{i \in \mathbb{Z}^n} (x_i^h)^\alpha \phi_i^h(x - x_j^h),$$

and therefore, using the definition of ξ_α^h and (3.5),

$$\begin{aligned} (x - x_j^h)^\alpha - \tilde{\mathcal{I}}_h(x - x_j^h)^\alpha & = \xi_\alpha^h(x - x_j^h) \\ & = \xi_\alpha^h(x), \text{ for } |\alpha| = k+1. \end{aligned} \tag{3.15}$$

Since, $\tilde{\mathcal{I}}_h p = p$ for $p \in \mathcal{P}^k$, we have

$$(x - x_j^h)^\alpha - \tilde{\mathcal{I}}_h(x - x_j^h)^\alpha = 0, \text{ for } |\alpha| \le k.$$

Therefore, using (3.13) and (3.15), we have

(3.16)

$$\begin{aligned} & u(x) - \tilde{\mathcal{I}}_h u(x) \\ & \quad = \sum_{|\alpha| \le k} \frac{1}{\alpha!}(D^\alpha u)(x_j^h)\big[(x - x_j^h)^\alpha - \tilde{\mathcal{I}}_h(x - x_j^h)^\alpha\big] \\ & \qquad + \sum_{|\alpha|=k+1} \frac{1}{\alpha!}(D^\alpha u)(x_j^h)\big[(x - x_j^h)^\alpha - \tilde{\mathcal{I}}_h(x - x_j^h)^\alpha\big] \\ & \qquad + R_{k+1}u(x) - \tilde{\mathcal{I}}_h R_{k+1}u(x) \\ & \quad = \sum_{|\alpha|=k+1} \frac{1}{\alpha!}(D^\alpha u)(x_j^h)\xi_\alpha^h(x) + R_{k+1}u(x) - \tilde{\mathcal{I}}_h R_{k+1}u(x), \text{ for all } x \in I_j^h. \end{aligned}$$

Thus,

$$\|u - \tilde{\mathcal{I}}_h u\|_{1,I_j^h} \le \| \sum_{|\alpha|=k+1} \frac{1}{\alpha!}(D^\alpha u)(x_j^h)\xi_\alpha^h(x)\|_{1,I_j^h}$$
$$+ \|R_{k+1}u\|_{1,I_j^h} + \|\tilde{\mathcal{I}}_h R_{k+1}u\|_{1,I_j^h}. \tag{3.17}$$

We will now find upper bounds for the terms in the right hand side of this inequality.

Since $R_{k+1}v = 0$ for all $v \in \mathcal{P}^{k+1}(I_j^h)$, using a Bramble-Hilbert argument (cf.[10]) one can prove that

$$\|R_{k+1}u\|_{1,I_j^h} \le Ch^{k+1} \sum_{|\alpha|=k+2} \|D^\alpha u\|_{q-1,I_j^h}. \tag{3.18}$$

From the definition of $\tilde{\mathcal{I}}_h$, we have for $x \in I_j^h$,

$$\tilde{\mathcal{I}}_h R_{k+1}u(x) = \sum_{l \in A_j^h} R_{k+1}u(x_l^h)\phi_l^h(x),$$

and thus from (2.2),

$$\|\tilde{\mathcal{I}}_h R_{k+1}u\|_{1,I_j^h} \le \|R_{k+1}u\|_{\infty,\tilde{B}_j^h} \sum_{l \in A_j^h} \|\phi_l^h\|_{1,I_j^h}$$
$$\le Ch^{\frac{n}{2}-1}\|R_{k+1}u\|_{\infty,\tilde{B}_j^h}. \tag{3.19}$$

Now using (3.14) and a scaled Sobolev Inequality, we have

$$\begin{aligned}
&|R_{k+1}u(x)| \\
&\quad \le (k+2) \sum_{|\alpha|=k+2} \|x - x_j^h\|^{|\alpha|} \int_0^1 \frac{s^{k+1}}{\alpha!} |(D^\alpha u)(x + s(x_j^h - x))|\, ds \\
&\quad \le Ch^{k+2} \sum_{|\alpha|=k+2} h^{-\frac{n}{2}} \|D^\alpha u\|_{q,\tilde{B}_j^h}, \text{ for all } x \in \tilde{B}_j^h,
\end{aligned}$$

and therefore,

$$\|R_{k+1}u\|_{\infty,\tilde{B}_j^h} \le Ch^{k+2-\frac{n}{2}} \sum_{|\alpha|=k+2} \|D^\alpha u\|_{q,\tilde{B}_j^h}.$$

Thus using the above inequality in (3.19), we get

$$\|\tilde{\mathcal{I}}_h R_{k+1}u\|_{1,I_j^h} \le Ch^{k+1} \sum_{|\alpha|=k+2} \|D^\alpha u\|_{q,\tilde{B}_j^h}. \tag{3.20}$$

Hence using (3.18) and (3.20) in (3.17) we get,

$$\|u - \tilde{\mathcal{I}}_h u\|_{1,I_j^h} \leq \left\| \sum_{|\alpha|=k+1} \frac{1}{\alpha!}(D^\alpha u)(x_j^h)\xi_\alpha^h(x) \right\|_{1,I_j^h} + Ch^{k+1} \sum_{|\alpha|=k+2} \|D^\alpha u\|_{q,\tilde{B}_j^h},$$

and therefore for any $\delta > 0$, we have

$$\|u - \tilde{\mathcal{I}}_h u\|^2_{1,I_j^h} \leq (1+\delta^2) \left\| \sum_{|\alpha|=k+1} \frac{1}{\alpha!}(D^\alpha u)(x_j^h)\xi_\alpha^h(x) \right\|^2_{1,I_j^h} + \left(1 + \frac{1}{\delta^2}\right) Ch^{2k+2} \sum_{|\alpha|=k+2} \|D^\alpha u\|^2_{q,\tilde{B}_j^h},$$

which is (3.11).

(b) Let $x \in I_j^h$. Then from (3.16), we have

$$\sum_{|\alpha|=k+1} \frac{1}{\alpha!}(D^\alpha u)(x_j^h)\xi_\alpha^h(x) = u(x) - \tilde{\mathcal{I}}_h u(x) - R_{k+1}u(x) + \tilde{\mathcal{I}}_h R_{k+1} u(x),$$

and therefore, using (3.18) and (3.20) we get

$$\begin{aligned} \left\| \sum_{|\alpha|=k+1} \frac{1}{\alpha!}(D^\alpha u)(x_j^h)\xi_\alpha^h(x) \right\|_{1,I_j^h} &\leq \|u - \tilde{\mathcal{I}}_h u\|_{1,I_j^h} + \|R_{k+1}u\|_{1,I_j^h} + \|\tilde{\mathcal{I}}_h R_{k+1} u\|_{1,I_j^h} \\ &\leq \|u - \tilde{\mathcal{I}}_h u\|_{1,I_j^h} + Ch^{k+1} \sum_{|\alpha|=k+2} \|D^\alpha u\|_{q,\tilde{B}_j^h}. \end{aligned}$$

(3.12) follows immediately from this result. □

Remark 3.5 In 1-d, we have a stronger result: if $u \in H^{k+2}(B_j^h)$, then

(a) for any $\delta > 0$,

$$\|u - \tilde{\mathcal{I}}_h u\|^2_{1,I_j^h} \leq (1+\delta^2) \frac{|u^{(k+1)}(x_j^h)|^2}{(k+1)!^2} \|\xi_{k+1}^h\|^2_{1,I_j^h} + \left(1 + \frac{1}{\delta^2}\right) Ch^{2(k+1)} |u|^2_{k+2,B_j^h}; \tag{3.21}$$

(b) for any $\delta > 0$,

$$\frac{|u^{(k+1)}(x_j^h)|^2}{(k+1)!^2}\|\xi_{k+1}^h\|_{1,I_j^h}^2 \leq (1+\delta^2)\|u - \tilde{\mathcal{I}}_h u\|_{1,I_j^h}^2 + \left(1+\frac{1}{\delta^2}\right) C h^{2(k+1)} |u|_{k+2,B_j^h}^2. \tag{3.22}$$

We now define certain sets, associated with the particles and Ω, which will be used in our next result, which is the main result of this section. Let I_j^h be the *cell* centered at x_j^h, as in Section 2. Recall from Section 2 that, corresponding to I_j^h, we defined $B_j^h = \{\cup_{m\in A_j^h} B_{Rh}(x_m^h)\} \cup I_j^h$, where $A_j^h = \{m \in \mathbb{Z}^n : \mathring{\eta}_m^h \cap I_j^h \neq \emptyset\}$ and $B_{Rh}(x_m^h)$ is the ball of radius Rh centered at x_m^h. We also recall that $B_j^h \subset \tilde{B}_j^h = B_{\tilde{R}h}(x_j^h)$. Define the following sets:

$$\begin{aligned}
\bar{\mathcal{A}}^h &= \{m \in \mathbb{Z}^n : \Omega \cap \mathring{I}_m^h \neq \emptyset\}, & \overline{\Omega}_h &= \cup_{j\in\bar{\mathcal{A}}^h} I_j^h \\
\underline{\mathcal{A}}^h &= \{m \in \mathbb{Z}^n : I_m^h \subset \Omega\}, & \underline{\Omega}_h &= \cup_{j\in\underline{\mathcal{A}}^h} I_j^h \\
\underline{B}^h &= \{\cup_{j\in\underline{\mathcal{A}}^h} \tilde{B}_j^h\} \cup \Omega, & \overline{B}^h &= \cup_{j\in\bar{\mathcal{A}}^h} \tilde{B}_j^h
\end{aligned}$$

It is clear that $\underline{\Omega}_h \subset \Omega \subset \overline{\Omega}_h$, and $|\Omega - \underline{\Omega}_h| \to 0$, $|\overline{\Omega}_h - \Omega| \to 0$ as $h \to 0$. Also $\Omega \subset \underline{B}^h \subset \overline{B}^h$, and $|\underline{B}^h - \Omega| \to 0$, $|\overline{B}^h - \Omega| \to 0$ as $h \to 0$. We also assume that R_0 is such that $\overline{B}^h \subset B_{R_0 - Rh}$.

We will now study the interpolation error $(u - \tilde{\mathcal{I}}_h u)$. An interpolation error estimate, $\|u - \tilde{\mathcal{I}}_h u\|_{1,\Omega} \approx O(h^k)$ was proved in [13,17] for the standard RKP shape functions; in fact, a more general result, namely, an estimate for $\|u - \tilde{\mathcal{I}}_h u\|_{W^{l,q}(\Omega)}$ was obtained in [13]. A similar order of convergence (for error in $H^{1,\infty}$ norm) was also obtained for MLS shape functions in [1,2]. We note that the definition of $\tilde{\mathcal{I}}_h u$ for the standard RKP shape functions and MLS shape functions differs slightly from our definition in (3.2). Using some of the arguments in the proof of the our next result, we will obtain an estimate of $\|u - \tilde{\mathcal{I}}_h u\|_{1,\Omega}$ in the context of translation invariant particle shape functions that are reproducing of order k. Moreover, our next result, which is of asymptotic nature, gives some information on the size of $\frac{\|u - \tilde{\mathcal{I}}_h u\|_{1,\Omega}}{h^k}$ which will help us to select "good" shape functions; this will be discussed in Section 4.

Theorem 3.1 *Suppose ϕ is reproducing of order k. Let $\bar{\lambda}$ be the largest eigenvalue of the matrix A given in (3.7). Suppose $q > \frac{n}{2}$ when $n \geq 2$, and $q = 0$ when $n = 1$. Then, we have*

$$\sup_{u\in H^{k+2+q}(\Omega)} \lim_{h\to 0} \frac{\|u - \tilde{\mathcal{I}}_h u\|_{1,\Omega}^2}{h^{2k} Q_h(u)} = \bar{\lambda}, \tag{3.23}$$

where

$$Q_h(u) = |u|^2_{k+1,\Omega} + h \sum_{|\alpha|=k+2} \|D^\alpha u\|^2_{q,\Omega}. \tag{3.24}$$

Note: In (3.23), we consider $u \in H^{k+2+q}(\Omega)$ such that $u \notin \mathcal{P}^k(\Omega)$. A sketch of the proof of Theorem 3.1 appeared in [8]. We here present the complete proof.

Proof. We will first prove that for $u \in H^{k+2+q}(\Omega)\backslash\mathcal{P}^k(\Omega)$,

$$\lim_{h\to 0} \frac{\|u - \tilde{\mathcal{I}}_h u\|^2_{1,\Omega}}{h^{2k} Q_h(u)} = \frac{\int_\Omega V^T(x) A V(x)\, dx}{|u|^2_{k+1,\Omega}}, \tag{3.25}$$

where

$$V^T(x) = [D^{\alpha(1)}u(x), D^{\alpha(2)}u(x), \dots, D^{\alpha(M_k)}u(x)]$$

and $\alpha(i)$, $1 \le i \le M_k$, are the multi-indices with $|\alpha(i)| = k+1$.

Let $u \in H^{k+2+q}(\Omega)$, and suppose $\bar{u} = Eu$ where E is the extension operator introduced in the beginning of this section. Since $\Omega \subset \overline{\Omega}_h$, we have

$$\|u - \tilde{\mathcal{I}}_h u\|^2_{1,\Omega} \le \|\bar{u} - \tilde{\mathcal{I}}_h \bar{u}\|^2_{1,\overline{\Omega}_h} = \sum_{j\in\bar{\mathcal{A}}^h} \|\bar{u} - \tilde{\mathcal{I}}_h \bar{u}\|^2_{1,I^h_j}.$$

Therefore, using (3.11), (3.6), and recalling that $\overline{B}^h = \cup_{j\in\bar{\mathcal{A}}^h} \tilde{B}^h_j$, we get for any $\delta > 0$,

$$\begin{aligned}
\|u - \tilde{\mathcal{I}}_h u\|^2_{1,\Omega} &\le (1+\delta^2) \sum_{j\in\bar{\mathcal{A}}^h} \left\| \sum_{|\alpha|=k+1} \frac{1}{\alpha!} (D^\alpha \bar{u})(x^h_j) \xi^h_\alpha(x) \right\|^2_{1,I^h_j} \\
&\quad + \left(1 + \frac{1}{\delta^2}\right) C h^{2k+2} \sum_{j\in\bar{\mathcal{A}}^h} \sum_{|\alpha|=k+2} \|D^\alpha \bar{u}\|^2_{q,\tilde{B}^h_j} \\
&\le \left(1+\delta^2\right) h^{2k} \sum_{j\in\bar{\mathcal{A}}^h} h^n V^T_j (A + h^2 B) V_j \\
&\quad + \left(1 + \frac{1}{\delta^2}\right) C h^{2k+2} \sum_{|\alpha|=k+2} \|D^\alpha \bar{u}\|^2_{q,\overline{B}^h},
\end{aligned} \tag{3.26}$$

where

$$V^T_j = [D^{\alpha(1)}\bar{u}(x^h_j), D^{\alpha(2)}\bar{u}(x^h_j), \dots, D^{\alpha(M_k)}\bar{u}(x^h_j)],$$

and A, B are the matrices in (3.7) and (3.8), respectively. Dividing (3.26) by $h^{2k} Q_h(u)$, where $Q_h(u)$ is defined in (3.24), we get

$$\frac{\|u - \tilde{\mathcal{I}}_h u\|^2_{1,\Omega}}{h^{2k} Q_h(u)} \le (1+\delta^2) \frac{\sum_{j \in \bar{\mathcal{A}}^h} h^n V_j^T (A + h^2 B) V_j}{Q_h(u)}$$

$$+(1+\frac{1}{\delta^2}) C h^2 \frac{\sum_{|\alpha|=k+2} \|D^\alpha \bar{u}\|^2_{q,\overline{B}^h}}{Q_h(u)}. \tag{3.27}$$

A typical term of the quadratic form $V_j^T (A + h^2 B) V_j$ is

$$D^{\alpha(i)} \bar{u}(x_j^h)(A_{il} + h^2 B_{il}) D^{\alpha(l)} \bar{u}(x_j^h).$$

Since

$$\lim_{h \to 0} \sum_{j \in \bar{\mathcal{A}}^h} h^n D^{\alpha(i)} \bar{u}(x_j^h) A_{il} D^{\alpha(l)} \bar{u}(x_j^h) = \int_\Omega D^{\alpha(i)} u(x) A_{il} D^{\alpha(l)} u(x)\, dx$$

and

$$\lim_{h \to 0} h^2 \sum_{j \in \bar{\mathcal{A}}^h} h^n D^{\alpha(i)} \bar{u}(x_j^h) B_{il} D^{\alpha(l)} \bar{u}(x_j^h) = 0,$$

we have,

$$\lim_{h \to 0} \sum_{j \in \bar{\mathcal{A}}^h} h^n V_j^T (A + h^2 B) V_j = \int_\Omega V^T(x) A V(x)\, dx. \tag{3.28}$$

Since $|\overline{B}^h - \Omega| \to 0$ as $h \to 0$, we have

$$\lim_{h \to 0} \sum_{|\alpha|=k+2} \|D^\alpha \bar{u}\|^2_{q,\overline{B}^h} = \sum_{|\alpha|=k+2} \|D^\alpha u\|^2_{q,\Omega}. \tag{3.29}$$

Also $\lim_{h \to 0} Q_h(u) = |u|_{k+1,\Omega}$. Thus, for any $\delta > 0$, using (3.28) and (3.29) in (3.27), we get

$$\limsup_{h \to 0} \frac{\|u - \tilde{\mathcal{I}}_h u\|^2_{1,\Omega}}{h^{2k} Q_h(u)} \le (1+\delta^2) \frac{\int_\Omega V^T(x) A V(x)\, dx}{|u|^2_{k+1,\Omega}},$$

and, since $\delta > 0$ is arbitrary, we have

$$\limsup_{h \to 0} \frac{\|u - \tilde{\mathcal{I}}_h u\|^2_{1,\Omega}}{h^{2k} Q_h(u)} \le \frac{\int_\Omega V^T(x) A V(x)\, dx}{|u|^2_{k+1,\Omega}}. \tag{3.30}$$

We next show that

$$\frac{\int_\Omega V^T(x) A V(x)\, dx}{|u|^2_{k+1,\Omega}} \le \liminf_{h \to 0} \frac{\|u - \tilde{\mathcal{I}}_h u\|^2_{1,\Omega}}{h^{2k} Q_h(u)}. \tag{3.31}$$

Consider a cell I_j^h, $j \in \underline{\mathcal{A}}^h$. From (3.12), we have for any $\delta > 0$,

$$\left\| \sum_{|\alpha|=k+1} \frac{1}{\alpha!} D^\alpha u(x_j^h) \xi_\alpha^h(x) \right\|_{1,I_j^h}^2 \leq (1+\delta^2)\|u - \tilde{\mathcal{I}}_h u\|_{1,I_j^h}^2 + \left(1 + \frac{1}{\delta^2}\right) C h^{2k+2} \sum_{|\alpha|=k+2} \|D^\alpha \bar{u}\|_{q,\tilde{B}_j^h}^2,$$

and using (3.6), we get

$$h^{2k+n} V_j^T (A + h^2 B) V_j \leq (1+\delta^2)\|u - \tilde{\mathcal{I}}_h u\|_{1,I_j^h}^2 + \left(1 + \frac{1}{\delta^2}\right) C h^{2k+2} \sum_{|\alpha|=k+2} \|D^\alpha \bar{u}\|_{q,\tilde{B}_j^h}^2.$$

Therefore, recalling that $\cup_{j \in \underline{\mathcal{A}}^h} I_j^h = \underline{\Omega}_h \subset \Omega$, we have

$$\begin{aligned} \sum_{j \in \underline{\mathcal{A}}^h} & h^{2k+n} V_j^T (A + h^2 B) V_j \\ &\leq (1+\delta^2)\|u - \tilde{\mathcal{I}}_h u\|_{1,\underline{\Omega}_h}^2 + \left(1 + \frac{1}{\delta^2}\right) C h^{2k+2} \sum_{|\alpha|=k+2} \sum_{j \in \underline{\mathcal{A}}^h} \|D^\alpha \bar{u}\|_{q,\tilde{B}_j^h}^2 \\ &\leq (1+\delta^2)\|u - \tilde{\mathcal{I}}_h u\|_{1,\Omega}^2 + \left(1 + \frac{1}{\delta^2}\right) C h^{2k+2} \sum_{|\alpha|=k+2} \|D^\alpha \bar{u}\|_{q,\underline{B}^h}^2. \end{aligned}$$

Hence, dividing both the sides by $h^{2k} Q_h(u)$, we get

$$\frac{\sum_{j \in \underline{\mathcal{A}}^h} h^n V_j^T (A + h^2 B) V_j}{Q_h(u)} \leq (1+\delta^2) \frac{\|u - \tilde{\mathcal{I}}_h u\|_{1,\Omega}^2}{h^{2k} Q_h(u)} + \left(1 + \frac{1}{\delta^2}\right) \times C h^2 \frac{\sum_{|\alpha|=k+2} \|D^\alpha \bar{u}\|_{q,\underline{B}^h}^2}{Q_h(u)}. \tag{3.32}$$

Since

$$\lim_{h \to 0} \sum_{j \in \underline{\mathcal{A}}^h} h^n V_j^T (A + h^2 B) V_j = \int_\Omega V^T(x) A V(x)\, dx \tag{3.33}$$

and

$$\lim_{h \to 0} \sum_{|\alpha|=k+2} \|D^\alpha \bar{u}\|_{q,\underline{B}^h}^2 = \sum_{|\alpha|=k+2} \|D^\alpha u\|_{q,\Omega}^2, \tag{3.34}$$

taking the limit as $h \to 0$ in (3.32), we get, for any $\delta > 0$,

$$\frac{\int_\Omega V^T(x)AV(x)\,dx}{|u|^2_{k+1,\Omega}} \le (1+\delta^2)\liminf_{h\to 0}\frac{\|u-\tilde{\mathcal{I}}_h u\|^2_{1,\Omega}}{h^{2k}Q_h(u)}$$

and therefore, we have (3.31). Now combining (3.30) and (3.31), we see that $\lim_{h\to 0}\frac{\|u-\tilde{\mathcal{I}}_h u\|^2_{1,\Omega}}{h^{2k}Q_h(u)}$ exists, and

$$\lim_{h\to 0}\frac{\|u-\tilde{\mathcal{I}}_h u\|^2_{1,\Omega}}{h^{2k}Q_h(u)} = \frac{\int_\Omega V^T(x)AV(x)\,dx}{|u|^2_{k+1,\Omega}},$$

which is (3.25).

Since $\bar{\lambda}$ is the largest eigenvalue of the matrix A, from the usual variational characterization of eigenvalues, we have

$$\int_\Omega V^T(x)AV(x)\,dx \le \bar{\lambda}\int_\Omega \sum_{i=1}^{M_k}|D^{\alpha(i)}u(x)|^2\,dx = \bar{\lambda}|u|^2_{k+1,\Omega}.$$

Thus from (3.25) we get

$$\lim_{h\to 0}\frac{\|u-\tilde{\mathcal{I}}_h u\|^2_{1,\Omega}}{h^{2k}Q_h(u)} \le \bar{\lambda}, \qquad \text{for any } u \in H^{k+2+q}(\Omega)\backslash\mathcal{P}^k(\Omega).$$

Hence

$$\sup_{u\in H^{k+2+q}(\Omega)}\lim_{h\to 0}\frac{\|u-\tilde{\mathcal{I}}_h u\|^2_{1,\Omega}}{h^{2k}Q_h(u)} \le \bar{\lambda}. \tag{3.35}$$

Let $\bar{v} = [v_1, v_2, \cdots, v_{M_k}]^T$ be an eigenvector of A corresponding to $\bar{\lambda}$. Then it is easily seen that there is a $u \in \mathcal{P}^{k+1}$ such that the vector $V(x) = \bar{v}$. For this particular u, we have

$$\frac{\int_\Omega V^T(x)AV(x)\,dx}{|u|^2_{k+1,\Omega}} = \bar{\lambda}.$$

Hence, from (3.35) we conclude that

$$\sup_{u\in H^{k+2+q}(\Omega)}\lim_{h\to 0}\frac{\|u-\tilde{\mathcal{I}}_h u\|^2_{1,\Omega}}{h^{2k}Q_h(u)} = \bar{\lambda},$$

which is the desired result. □

Remark 3.6 We know from (3.2) that the interpolant $\tilde{\mathcal{I}}_h u$ depends on the extension $\bar{u}$ of u. But it is clear from the proof of Theorem 3.1 that (3.23) is valid for any extension $\bar{u} \in H^{k+2+q}(B_{R_0})$.

Remark 3.7 We note that same result holds for the H^1-seminorm of the interpolation error, *i.e.,* for $q > \frac{n}{2}$ when $n \geq 2$, and $q = 0$ when $n = 1$,

$$\sup_{u \in H^{k+2+q}(\Omega)} \lim_{h \to 0} \frac{|u - \tilde{\mathcal{I}}_h u|^2_{1,\Omega}}{h^{2k}[|u|^2_{k+1,\Omega} + h \sum_{|\alpha|=k+2} \|D^\alpha u\|^2_{q,\Omega}]} = \bar{\lambda}.$$

The proof is similar to the proof of Theorem 3.1. We further note that it is possible to obtain a result involving $|u - \tilde{\mathcal{I}}_h u|_{s,\Omega}$, for $1 < s \leq k+1$. We have not included a proof of this result in this paper.

Remark 3.8 A result, similar to (3.23), also holds for the L_2-norm of the interpolation error; it is stated as follows: For $q > \frac{n}{2}$ when $n \geq 2$, and $q = 0$ when $n = 1$

$$\sup_{u \in H^{k+2+q}(\Omega)} \lim_{h \to 0} \frac{\|u - \tilde{\mathcal{I}}_h u\|^2_{0,\Omega}}{h^{2(k+1)}[|u|^2_{k+1,\Omega} + h \sum_{|\alpha|=k+2} \|D^\alpha u\|^2_{q,\Omega}]} = \bar{\mu},$$

where $\bar{\mu}$ is the largest eigenvalue of the matrix B, defined in (3.8). The proof of this result is parallel to the proof of Theorem 3.1.

Remark 3.9 From (3.27) in the proof of Theorem 3.1, we can obtain an interpolation error estimate. We briefly show this here. We first note that, using Sobolev's inequality, (3.3), and recalling that $\overline{B}^h \subset B_{R_0}$, we can show that

$$\sum_{j \in \bar{\mathcal{A}}^h} h^n V_j^T (A + h^2 B) V_j \leq C \sum_{|\alpha|=k+1} \|D^\alpha \bar{u}\|^2_{\infty, \overline{B}^h} \leq C \|u\|^2_{k+1+q,\Omega}.$$

Also from (3.3), we have

$$\sum_{|\alpha|=k+2} \|D^\alpha \bar{u}\|^2_{q, \overline{B}^h} \leq C \|u\|^2_{k+2+q,\Omega}.$$

Now, using these inequalities in (3.27) with $\delta = 1$, and cancelling $Q_h(u)$ we get

$$\|u - \tilde{\mathcal{I}}_h u\|_{1,\Omega} \leq C h^k \|u\|_{k+2+q,\Omega}, \tag{3.36}$$

where C may depend on Ω, but is independent of u and h. Similarly, we can show (cf. Remark 3.8) that

$$\|u - \tilde{\mathcal{I}}_h u\|_{0,\Omega} \leq C h^{k+1} \|u\|_{k+2+q,\Omega}.$$

It is also possible to obtain interpolation error estimates under a weaker regularity assumption on u. Replacing k by $k-1$ in (3.13) and using some of the arguments in the proof of Theorem 3.1, we can show that

$$\|u - \tilde{\mathcal{I}}_h u\|_{l,\Omega} \leq C h^{k+1-l} \|u\|_{k+1+q,\Omega}, \quad l = 0, 1, \tag{3.37}$$

where C may depend on Ω, but is independent of u and h. We note, however, that these are not optimal error estimates, but they give the correct order of convergence. We note that the ideas used to prove (3.37) were also used in [13] in the context of standard RKP shape functions.

Remark 3.10 From Theorem 3.1 we know that

$$\lim_{h\to 0} \frac{\|u - \tilde{\mathcal{I}}_h u\|_{1,\Omega}^2}{h^{2k} Q_h(u)} \le \bar{\lambda}, \qquad \text{for all } u \in H^{k+2+q}(\Omega),$$

which can be written

$$\frac{\|u - \tilde{\mathcal{I}}_h u\|_{1,\Omega}^2}{h^{2k} Q_h(u)} \le \bar{\lambda}[1 + o(1)],$$

where $o(1) = o_u(1)$ depends on u. Hence

$$\begin{aligned} \|u - \tilde{\mathcal{I}}_h u\|_{1,\Omega}^2 &\le \bar{\lambda}[1 + o(1)]h^{2k} Q_h(u) \\ &\le \bar{\lambda} h^{2k} |u|_{k+1,\Omega}^2 + o(h^{2k}) \|u\|_{k+2+q,\Omega}^2. \end{aligned} \tag{3.38}$$

The right-hand side of (3.38) is written as an asymptotic formula. The importance of Theorem 3.1 is that it gives the exact form for the dominant term. We immediately see the relation between (3.38) and the error estimate (3.36) in Remark 3.9; the constant C in the error estimate would be a number larger than $\bar{\lambda}$. So, assuming the inequality in (3.38) is nearly an equality for small h, we see that $\bar{\lambda}$ provides an accurate indication of the interpolation error.

Remark 3.11 For $u \in H^{k+2+q}(\Omega)$, one can show by carefully following the proof of (3.25) in the proof of Theorem 3.1, with $Q_h(u)$ replaced by $Q(u) = |u|_{k+1,\Omega}^2 + \sum_{|\alpha|=k+2} \|D^\alpha u\|_{q,\Omega}^2$, that

$$\lim_{h\to 0} \frac{\|u - \tilde{\mathcal{I}}_h u\|_{1,\Omega}^2}{h^{2k}} = \int_\Omega V^T(x) A V(x)\, dx, \tag{3.39}$$

where $V^T(x) = [D^{\alpha(1)}u(x), D^{\alpha(2)}u(x), \ldots, D^{\alpha(M_k)}u(x)]$, and $\alpha(i), 1 \le i \le M_k$ are multi-indices with $|\alpha(i)| = k + 1$. In fact, in 1-d, the matrix A is a 1×1 matrix, $A = |\xi_{k+1}|_{1,(0,1)}^2/(k + 1)!^2$. Thus in 1-d, we get

$$\|u - \tilde{\mathcal{I}}_h u\|_{1,\Omega} = \frac{|\xi_{k+1}|_{1,(0,1)}}{(k+1)!} h^k |u|_{k+1,\Omega} + o(1)h^k. \tag{3.40}$$

The same is true if the left hand side of the above equality is replaced with the semi-norm $|u - \tilde{\mathcal{I}}_h u|_{1,\Omega}$.

We now prove a saturation theorem. Such theorems play an important role in approximation theory, and they can also serve as a basis for the verification of the implementation codes, as $h \to 0$.

Theorem 3.2 *Consider shape functions that are strongly r-reproducing of order k with $r = 1$. Let u be sufficiently smooth in Ω and suppose*

$$\|u - \tilde{\mathcal{I}}_h u\|_{1,\Omega} \leq Ch^{k+\epsilon}, \quad \epsilon > 0,$$

where $C = C_u$ may depend on u but is independent of h. Then u is a polynomial of degree k in Ω.

Proof. For a smooth function u, we know from (3.39) that

$$\frac{\|u - \tilde{\mathcal{I}}_h u\|_{1,\Omega}^2}{h^{2k}} = \int_\Omega V^T(x) A V(x)\, dx + o(1),$$

where $V^T(x) = [D^{\alpha(1)}u(x), D^{\alpha(2)}u(x), \ldots, D^{\alpha(M_k)}u(x)]$ and $\alpha(i)$, $1 \leq i \leq M_k$ are the multi-indices such that $|\alpha(i)| = k + 1$. Now, if u is such that $\|u - \tilde{\mathcal{I}}_h u\|_{1,\Omega} \leq Ch^{k+\epsilon}$, then from the above equality we have

$$\int_\Omega V^T(x) A V(x)\, dx = 0. \tag{3.41}$$

From the variational characterization of eigenvalues we have

$$\underline{\lambda} |u|_{k+1,\Omega}^2 \leq \int_\Omega V^T(x) A V(x)\, dx,$$

where $\underline{\lambda}$ is the smallest eigenvalue of A. Since the shape functions are strongly r-reproducing of order k with $r = 1$, we know, as observed in Remark 3.3, that $\underline{\lambda} > 0$. Thus from (3.41), we get $|u|_{k+1,\Omega} = 0$, which implies that u is a polynomial of degree k in Ω. □

We note that, for strongly r-reproducing shape functions of order k with $r = 0$, we can prove an analogous saturation theorem for $\|u - \tilde{\mathcal{I}}_h u\|_{0,\Omega}$, which we have not presented in this paper. We further note that strongly r-reproducing shape functions of order k with $r = s$ will be required to prove a saturation theorem for $\|u - \tilde{\mathcal{I}}_h u\|_{s,\Omega}$, $1 < s \leq k + 1$; we have not included such results in this paper.

4 Selection of shape functions

We have seen in the Section 3 (cf. Remark 3.9) that if the translation invariant particle shape functions are reproducing of order k, then for a smooth function u,

$$\|u - \tilde{\mathcal{I}}_h u\|_{1,\Omega} \leq Ch^k,$$

where $\tilde{\mathcal{I}}_h u$ is the interpolant of u, as defined in (3.2) in terms of these shape functions. There are many classes of translation invariant shape functions that are reproducing of order k. We saw in Section 2 that translation invariant RKP shape functions depend on the weight function $\omega(x)$, and different choices of $\omega(x)$ generate different classes of RKP shape functions that are reproducing of order k. Suppose we have finitely many classes of shape functions, which are reproducing of order k, at our disposal. Then it is important to select among them a particular class of shape functions that will yield small values for $\|u - \tilde{\mathcal{I}}_h u\|_{1,\Omega}$, for a wide class of functions u.

From (3.23) (also see Remark 3.10), we know that for all $u \in H^{k+2+q}(\Omega)$ with $q > n/2$,

$$R_h(u,\phi) \equiv \frac{\|u - \tilde{\mathcal{I}}_h u\|_{1,\Omega}}{h^k \sqrt{Q_h(u)}} \lessapprox \sqrt{\bar{\lambda}}, \text{ for small } h, \tag{4.1}$$

where the interpolant $\tilde{\mathcal{I}}_h u$ is in terms of the shape functions corresponding to the basic shape function ϕ, and $\bar{\lambda}$ is the largest eigenvalue of the matrix A given in (3.7). From Remark 3.7, we can also replace the numerator of the left-hand side of (4.1) by $|u - \tilde{\mathcal{I}}_h u|_{1,\Omega}$. We note that $\bar{\lambda}$ depends only on the *basic shape function* $\phi(x)$; we write $\bar{\lambda}(\phi)$ to indicate this dependence. We emphasize that $\bar{\lambda}(\phi)$ does not depend on u or on h. Moreover, $\bar{\lambda}(\phi)$ is computable. We will use (4.1) to rank the approximation qualities of various classes of translation invariant shape functions. Specifically, given a finite collection of different classes of shape functions, we will rank them according to the size of $\bar{\lambda}(\phi)$, and we will choose the class that yields the smallest value of $\bar{\lambda}(\phi)$.

Remark 4.1 Let $S = \{\phi\}$ be a finite collection of basic shape functions that are reproducing of order k, and let $\phi^* \in S$ be such that $\bar{\lambda}(\phi^*) < \bar{\lambda}(\phi)$ for all $\phi \in S$, $\phi \neq \phi^*$. The selection process, described above, does not guarantee that, for each $u \in H^{k+2+q}(\Omega)$ with $q > n/2$, $R_h(u,\phi^*) \leq R_h(u,\phi)$ for all $\phi \in S$. It only ensures that $R_h(u,\phi^*) \lessapprox \sqrt{\bar{\lambda}(\phi^*)}$ for any $u \in H^{k+2+q}(\Omega)$ for sufficiently small h. Moreover, there exists $u_0 \in H^{k+2+q}(\Omega)$ such that $\sqrt{\bar{\lambda}(\phi^*)} < R_h(u_0,\phi)$ for h small enough.

We now show this ranking numerically in 1-d. We note that in 1-d, $\bar{\lambda} = \left(\frac{|\xi_{k+1}|_{1,(0,1)}}{(k+1)!}\right)^2$. In the rest of this paper, we suppress $(0,1)$ in $|\xi_{k+1}|_{1,(0,1)}$ and instead write $|\xi_{k+1}|_1$. We have considered $\omega(x)$ given in (2.13) with $\delta = 2$, (2.14), and (2.15) with $l = 2, 4$, and constructed the corresponding translation invariant RKP shape functions that are reproducing of order $k = 1$. We then computed the quantity $|\xi_2|_1$, corresponding to each class of shape functions (for these choices of $\omega(x)$), for different values of R. The results are shown in Figure 4.1.

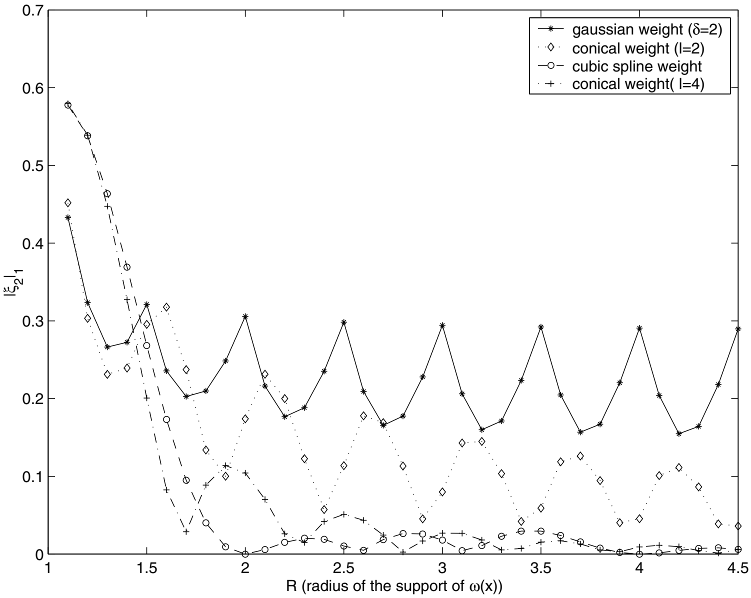

Fig. 4.1. The plot of $|\xi_2|_1$ with respect to the radius of the support, R, of the weight function $\omega(x)$. Four different weight functions have been used

From Figure 4.1, we infer the following regarding the interpolation of smooth functions by translation invariant RKP shape functions that are reproducing of order 1:

- The value of $|\xi_2|_1$ depends on the value of R, and therefore the selection of shape functions should depend on R.
- For a fixed value of R, we compare the values $|\xi_2|_1$, and choose the shape function that leads to the smallest value of $|\xi_2|_1$. For example, when $R = 1.7$, the values of $|\xi_2|_1$ are $0.237, 0.203, 0.095$, and 0.029 for $\omega(x)$ given by (2.15) with $l = 2$, (2.13) with $\delta = 2$, (2.14), and (2.15) with $l = 4$, respectively. Therefore, when $R = 1.7$, we propose to choose shape functions corresponding to the conical weight function with $l = 4$. Similarly, when $R = 2$, we propose to choose shape functions corresponding to the cubic spline weight function; this choice is clearly indicated in Figure 4.1.

To validate our proposal for the selection of the shape functions, we have considered the function $u(x) = x^4$ on the interval $\Omega = (0, 1)$ and computed the error $|u - \tilde{\mathcal{I}}_h u|_{1,\Omega}$, where $\tilde{\mathcal{I}}_h u$ is the interpolant of u as defined in (3.2) with $h = 1/n$, $n = 40, 50, \ldots, 100$. We recall that the definition of $\tilde{\mathcal{I}}_h u$ requires the values of $u(x)$ outside the interval Ω. First, we consider the natural analytic extension of u, *i.e.,* consider u to be x^4 outside Ω. We have considered the translation invariant RKP shape functions ϕ_i^h corresponding to the weight functions $\omega(x)$ used in Figure 4.1. We summarize the results in Table 4.1.

Table 4.1. The H^1-seminorm of the error, $|u - \tilde{\mathcal{I}}_h u|_{1,\Omega}$, where $\tilde{\mathcal{I}}_h u$ is the interpolant of $u(x) = x^4$ using the translation invariant shape functions that are reproducing of order 1, corresponding to various weight functions $\omega(x)$. The radius of support of $\omega(x)$ is $R = 1.7$

n	$\|u - \tilde{\mathcal{I}}_h u\|_{1,\Omega}$			
	Conical: $l = 2$	Gauss: $\delta = 2$	Cubic Spline	Conical: $l = 4$
40	1.607e-2	1.376e-2	6.435e-3	2.283e-3
50	1.281e-2	1.096e-2	5.130e-3	1.730e-3
60	1.066e-2	9.112e-3	4.267e-3	1.396e-3
70	9.126e-3	7.800e-3	3.653e-3	1.172e-3
80	7.980e-3	6.819e-3	3.194e-3	1.012e-3
90	7.090e-3	6.058e-3	2.838e-3	8.908e-4
100	6.379e-3	5.449e-3	2.553e-3	7.962e-4

From Table 4.1 and Figure 4.1, it is clear that the error $|u - \tilde{\mathcal{I}}_h u|_{1,\Omega}$ can be ranked according to the size of $|\xi_2|_1$ for the four choices of $\omega(x)$ considered here with $R = 1.7$; the error and $|\xi_2|_1$ are both minimum when $\omega(x)$ is the conical weight function with $l = 4$.

From (3.40) of Remark 3.11 (replacing $\|u - \tilde{\mathcal{I}}_h u\|_{1,\Omega}$ by $|u - \tilde{\mathcal{I}}_h u|_{1,\Omega}$), we know that for $\Omega = (0, 1)$ and $k = 1$,

$$\lim_{h\to 0} \frac{|u - \tilde{\mathcal{I}}_h u|_{1,\Omega}}{h|u|_{2,\Omega}} = \frac{|\xi_2|_1}{2!} \tag{4.2}$$

for all smooth function u. A simple calculation of the ratio $\kappa = \frac{2|u - \tilde{\mathcal{I}}_h u|_{1,\Omega}}{h|u|_{2,\Omega}}$, using the values in the Table 4.1, shows that $\kappa \to |\xi_2|_1$ as h gets smaller, for each $\omega(x)$ considered in Table 4.1, which illustrates (4.2).

We know that $\tilde{\mathcal{I}}_h u$ depends on the extension of u outside the interval $\Omega = (0, 1)$. We now extend $u = x^4$ outside Ω to a small neighborhood of Ω as follows:

$$u(x) = \begin{cases} 0, & x \le 0, \\ x^4, & 0 \le x \le 1, \\ g(x), & x \ge 1, \end{cases} \tag{4.3}$$

where

$$g(x) = \frac{130}{3} e^{(1-x)} - 104 e^{2(1-x)} + 86 e^{3(1-x)} - \frac{73}{3} e^{4(1-x)}.$$

We note that $u^{(3)}(x)$ of the extended function $u(x)$ is continuous. We have computed $|u - \tilde{\mathcal{I}}_h u|_{1,\Omega}$ where $u = x^4$ has been extended as (4.3) outside $\Omega = (0, 1)$ to a small neighborhood of Ω.

Table 4.2. The H^1-seminorm of the error, $|u - \tilde{\mathcal{I}}_h u|_{1,\Omega}$, where $\tilde{\mathcal{I}}_h u$ is the interpolant of $u(x) = x^4$ using the translation invariant shape functions that are reproducing of order 1, corresponding to different weight functions $w(x)$. The radius of support of $w(x)$ is $R = 1.7$. The function u has been extended outside $\Omega = (0, 1)$ as in (4.3)

n	$\lvert u - \tilde{\mathcal{I}}_h u\rvert_{1,\Omega}$			
	Conical: $l = 2$	Gauss: $\delta = 2$	Cubic Spline	Conical: $l = 4$
40	1.606e-2	1.375e-2	6.437e-3	2.278e-3
50	1.281e-2	1.096e-2	5.131e-3	1.728e-3
60	1.066e-2	9.112e-3	4.267e-3	1.395e-3
70	9.125e-3	7.800e-3	3.653e-3	1.172e-3
80	7.979e-3	6.819e-3	3.194e-3	1.012e-3
90	7.090e-3	6.058e-3	2.838e-3	8.906e-4
100	6.379e-3	5.449e-3	2.553e-3	7.960e-4

We see that the values of $|u - \tilde{\mathcal{I}}_h u|_{1,\Omega}$ in the Table 4.2 are very close to the corresponding values in the Table 4.1, and hence, the particular extension of u outside $\Omega = (0, 1)$ does not affect the error significantly. This observation illustrates Remark 3.6. In the rest of the numerical experiments in this paper, we will always analytically extend the function u outside Ω, to a small neighborhood of Ω, when computing $\tilde{\mathcal{I}}_h u$.

Our proposal of ranking and selecting shape functions is based on Theorem 3.1, and thus on the interpolation of smooth functions. But we have observed from further computations (not presented here) that the same selection process, based on the size of $|\xi_2|_1$, is also valid for the L_2 and H^1 projections. A detailed analysis of the selection procedure with respect to these projections will be addressed in a forthcoming paper.

So far we have shown that our process of selection and ranking of shape functions, which is based on the size of $|\xi_2|_1$, is successful when we use translation invariant shape functions. It is of interest to test this process on shape functions that are not translation invariant, *e.g.,* standard RKP shape functions that are defined by (2.7), (2.11), (2.9), and (2.12). We recall that our proposed selection process is based on Theorem 3.1 where translation invariant shape functions were used. We present the error $|u - \tilde{\mathcal{I}}_h u|_{1,\Omega}$ where $u(x) = x^4$ is defined on $\Omega = (0, 1)$, and $\tilde{\mathcal{I}}_h u(x)$ is the interpolant of u using standard RKP shape functions that reproduce polynomials of degree $k = 1$. These shape functions correspond to the weight functions considered in Figure 4.1.

The Table 4.3 and Figure 4.1 clearly shows that our procedure of selection of the shape function is also valid for standard RKP shape functions. Also a comparison of Tables 4.1 and 4.3 shows that the values in Table 4.3

Table 4.3. The H^1-seminorm of the error, $|u - \tilde{\mathcal{I}}_h u|_{1,\Omega}$, where $\tilde{\mathcal{I}}_h u$ is the interpolant of $u(x) = x^4$ using the standard RKP shape functions that are reproducing of order 1, corresponding to various weight functions $w(x)$. The radius of support of $w(x)$ is $R = 1.7$

$n = \frac{1}{h}$	$\lvert u - \tilde{\mathcal{I}}_h u\rvert_{1,\Omega}$			
	Conical: $l = 2$	Gauss: $\delta = 2$	Cubic Spline	Conical: $l = 4$
40	1.906e-2	1.749e-2	1.139e-2	9.739e-3
50	1.480e-2	1.345e-2	8.480e-3	7.023e-3
60	1.207e-2	1.089e-2	6.688e-3	5.379e-3
70	1.018e-2	9.132e-3	5.487e-3	4.295e-3
80	8.801e-3	7.855e-3	4.633e-3	3.535e-3
90	7.745e-3	6.886e-3	3.998e-3	2.979e-3
100	6.914e-3	6.127e-3	3.508e-3	2.557e-3

are bigger than the corresponding values in Table 4.1, suggesting that the performance of translation invariant RKP shape functions may be better than the performance of standard RKP shape functions. In fact, we will show in Section 6 that standard RKP shape functions give rise to a boundary layer, where as translation invariant shape functions do not show such behavior.

Thus we conclude that if we have to choose a basic shape function from a collection of basic shape functions that are reproducing of order k, we should choose the basic shape function that yields the smallest value of $|\xi_{k+1}|_1$. Our computations suggest that such a choice of basic shape function (and the corresponding translation invariant particle shape functions) will give the smallest interpolation error $|u - \tilde{\mathcal{I}}_h u|_{1,\Omega}$ among all the available shape functions. Moreover, the computations also suggest that, for uniformly distributed particles, the standard RKP shape functions can also be selected based on the size of $|\xi_{k+1}|_1$. We note that, though the computational examples presented in this section are 1-dimensional, the selection process is also valid in higher dimensions since the results in Section 3 are true in $\mathbb{R}^n$.

5 Approximation by translation invariant shape functions

We have seen in Section 3 (cf. Remark 3.9) that for u smooth,

$$\|u - \tilde{\mathcal{I}}_h u\|_{l,\Omega} \le C h^{k+1-l}, \quad l = 0, 1, \tag{5.1}$$

where $\tilde{\mathcal{I}}_h u$ is the interpolant of u, defined using particle shape functions ϕ_i^h that are reproducing of order k. From this interpolation error estimate, one immediately obtains an approximation error estimates:

$$\inf_{\phi \in S_h} \|u - \phi\|_{l,\Omega} \le \|u - \tilde{\mathcal{I}}_h u\|_{l,\Omega} \le C h^{k+1-l}, \quad l = 0, 1, \tag{5.2}$$

where $S_h = \text{span}\{\phi_i^h\}$, $i \in A^h = \{i \in \mathbb{Z}^n : \mathring{\eta}_i^h \cap \Omega \neq \emptyset\}$. Such approximation error estimates are important in assessing the accuracy of approximate solutions of boundary value problems obtained from the Galerkin method using particle shape functions, under the assumption that all the integrals in the Galerkin method are evaluated exactly. In this paper, we did not study the effect of the perturbation of these integrals (by numerical integration) on the accuracy of approximate solutions.

We note, however, that the interpolation error estimate – which is based on the reproducing property of shape functions – may give pessimistic estimate for the approximation error. Specifically, the order of convergence of the approximation error, as given in (5.2), may be lower than the actual order of convergence. In this section, we will show that the appropriate hypothesis for the approximation error estimate is the quasi-reproducing property of shape functions; this hypothesis gives the correct order of convergence for the approximation error. We will also show that the interpolation error may decrease at a higher rate, in the *pre-asymptotic range*, than is predicted by (5.1). This pre-asymptotic range can be so large that for practical accuracy, the asymptotic range is not visible.

The theory of approximation of functions by translation invariant shape functions was developed in [4], [23], [24] using Fourier Transform. An approximation result with respect to shape functions, which are not translation invariant, was proved in [8]. We cite a theorem from [24].

Theorem 5.1 *Suppose $\phi \in H^q(\mathbb{R}^n)$ has compact support. Then the following three conditions are are equivalent:*

1.

$$\hat{\phi}(0) \neq 0$$

and

$$D^\alpha \hat{\phi}(2\pi j) = 0, \textit{ for } 0 \neq j \in \mathbb{Z}^n \textit{ and } |\alpha| \leq k.$$

2. For $|\alpha| \leq k$,

$$\sum_{j \in \mathbb{Z}^n} j^\alpha \phi(x - j) = \lambda x^\alpha + q_{|\alpha|-1}(x),$$

$$\textit{where } \lambda \neq 0, \textit{ and degree } q_{|\alpha|-1} < |\alpha|. \tag{5.3}$$

The equality in (5.3) is for almost all $x \in \mathbb{R}^n$. The function of the right-hand side of (5.3) is, of course, continuous. If the function on the left-hand side is continuous, which will be the case if $q > n/2$, then (5.3) will hold for all $x \in \mathbb{R}^n$.

3. *For each $u \in H^{k+1}(\mathbb{R}^n)$, there are weights $w_j^h \in \mathbb{R}$, for $j \in \mathbb{Z}^n$ and $0 < h$, such that*

$$\left\| u - \sum_{j\in\mathbb{Z}^n} w_j^h \phi_j^h \right\|_{H^s(\mathbb{R}^n)} \tag{5.4}$$
$$\leq C h^{k+1-s} \|u\|_{H^{k+1}(\mathbb{R}^n)}, \quad \textit{for} \quad 0 \leq s \leq \min\{q, k+1\},$$

and

$$h^n \sum_{j\in\mathbb{Z}^n} (w_j^h)^2 \leq K^2 \|u\|^2_{H^0(\mathbb{R}^n)}.$$

Here C and K may depend on q, k, and s, but are independent of u and h. The exponent $k+1-s$ is the best possible if k is the largest integer for which (5.3) holds.

Remark 5.1 We note that (5.3) is precisely the definition of quasi-reproducing of order k (for the basic shape function $\phi(x)$; see (2.3)), which is equivalent to (2.4).

Remark 5.2 For a function $u \in H^{k+1}(\Omega)$, where $\Omega \in \mathbb{R}^n$ is a bounded domain with Lipschitz-continuous boundary, we know from Section 3 that u can be extended to a function $\bar{u} \in H^{k+1}(\mathbb{R}^n)$ satisfying (3.3). Thus for translation invariant shape functions that are reproducing of order k, (5.4) is true for $\bar{u}$ (with u replaced by $\bar{u}$). Then using (3.3) one can show that (5.4) is also true with $H^s(\mathbb{R}^n)$ and $H^{k+1}(\mathbb{R}^n)$ replaced by $H^s(\Omega)$ and $H^{k+1}(\Omega)$, respectively.

Now, from the definition of reproducing and quasi-reproducing shape functions, it is clear that if a basic shape function $\phi(x)$ is reproducing of order k, then it is also quasi-reproducible of order k. But a basic shape shape function may be reproducing of order k, but quasi-reproducing of higher order $k+k'$, $k' \geq 1$. In this case, the approximation error estimate is

$$\inf_{\phi\in S_h} \|u-\phi\|_{l,\Omega} \leq C h^{k+k'+1-l}, \quad l = 0, 1, \tag{5.5}$$

as seen from (5.4) and Remark 5.2. But (5.2), which is based on the interpolation error estimate (that uses the reproducing property of shape functions), only yields $O(h^k)$, and thus gives sub-optimal order of convergence of the approximation error.

We will illustrate this phenomenon with a numerical experiment. We consider the basic shape function $\phi(x)$, whose Fourier Transform is given by

$$\hat{\phi}(\xi) = (\sin(\xi/2)/(\xi/2))^4.$$

$\phi(x)$ is the B-spline of order 4 and is piecewise cubic with support $[-2, 2]$. It is reproducing of order $k = 1$, but quasi-reproducing of order 3. We consider the shape functions corresponding to this basic shape function. We computed the error $\|u - P_h^0 u\|_{0,\Omega}$ and $\|u - \tilde{\mathcal{I}}_h u\|_{0,\Omega}$ for $n = 1/h = 10, 12, 14, \ldots, 30$, where $u(x) = x^4$ and $\Omega = (0, 1)$. Here $P_h^0 u$ is the L_2-projection of u onto the space spanned by these shape functions, defined as

$$\begin{aligned} &P_h^0 u(x) \in \operatorname{span}\{\phi_i^h\}_{i \in A^h}, \\ &\int_0^1 (u - P_h^0 u)\phi_i^h \, dx = 0, \quad i \in A^h, \end{aligned} \tag{5.6}$$

where $A^h = \{m \in \mathbb{Z} : \mathring{\eta}_m^h \cap \Omega \neq \emptyset\}$. We note that the computation of $P_h^0 u$ does not require the values of $u(x)$ outside the domain Ω, but the set A^h contains some particles outside Ω. As before, $\tilde{\mathcal{I}}_h u$ is the interpolant of u, as defined in (3.2), with respect to these shape functions.

We have plotted loglog graphs of $E_{Int}^h = \|u - \tilde{\mathcal{I}}_h u\|_{0,\Omega}$ and $E_{Proj}^h = \|u - P_h^0 u\|_{0,\Omega}$ with respect to $n = 1/h$ in Figures 5.1(a) and 5.2(b).

We observe from Figure 5.1 that the values of $\|u - P_h^0 u\|_{0,\Omega}$ are much smaller than the values of $\|u - \tilde{\mathcal{I}}_h u\|_{0,\Omega}$. We also clearly see that $\|u - \tilde{\mathcal{I}}_h u\|_{0,\Omega} = O(h^2)$ and $\|u - P_h^0 u\|_{0,\Omega} = O(h^4)$. These rates are predicted by our analysis.

Also, since the particular basic shape function (B-spline of order 4) we considered for this experiment is quasi-reproducing of order 3 and reproducing of order 1, from (5.5) with $l = 0$ we expect $\|u - P_h^0 u\|_{0,\Omega} = O(h^4)$, whereas from (5.1) with $l = 0$ we expect $\|u - \tilde{\mathcal{I}}_h u\|_{0,\Omega} = O(h^2)$.

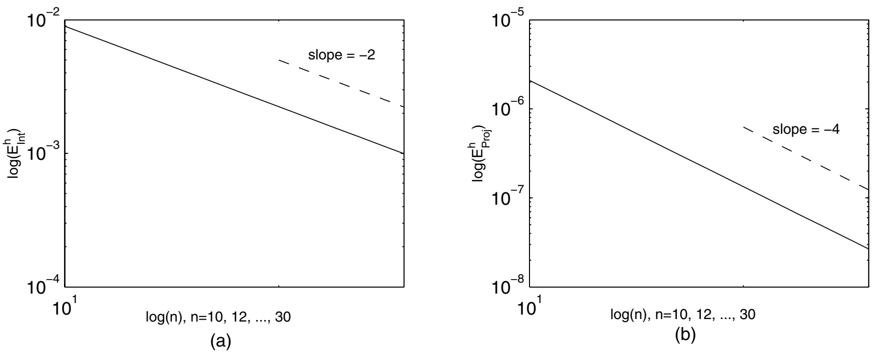

Fig. 5.1 (a), (b). The loglog graph of $E_{Int}^h = \|u - \tilde{\mathcal{I}}_h u\|_{0,\Omega}$ and $E_{Proj}^h = \|u - P_h^0 u\|_{0,\Omega}$ with respect to $n = 1/h$. $P_h^0 u$ is the L_2-projection of $u(x) = x^4$ onto the space spanned by shape functions that are reproducing of order 1 and quasi-reproducing of order 3. $\tilde{\mathcal{I}}_h u$ is the interpolant of $u(x)$ using the same shape functions

Computation of H^1 semi-norm of these errors shows that $|u-\tilde{\mathcal{I}}_h u|_{1,\Omega} = O(h)$ and $|u - P_h^0 u|_{1,\Omega} = O(h^3)$; we have not included these computations here. This indicates that (5.2), which is based on interpolation and the reproducing property of shape functions, may give pessimistic order of convergence for the approximation error. But (5.5), which is based on Theorem 5.1 and quasi-reproducing property of shape functions, yields higher order of convergence for the approximation error. We note that the order of convergence in the interpolation error estimate (5.1) is not pessimistic; it depends on the reproducing property of the shape functions, but is insensitive to the quasi-reproducing property of the shape functions, as was shown in this computational example. Thus an approximation error estimate, derived from the interpolation error estimate, *e.g.*, (5.2), may give pessimistic order of convergence when quasi-reproducing shape functions of higher order are used.

We note, however, that a basic shape function $\phi(x)$ may be reproducible of order k, and not quasi-reproducible of any higher order. In this case, Theorem 5.1 and (5.2) give the same order of convergence, *i.e.,* approximation error analysis using interpolation does not give pessimistic results. But even in this case, the H^1-norm (or H^1-seminorm) of the interpolation error may decrease at a higher rate than given in (5.1), with $l = 1$, in the pre-asymptotic range.

This phenomenon occurs when the basic shape function $\phi(x)$ is "almost quasi-reproducing" of order $(k+1)$, which we define as follows:

$$\text{(5.7)} \qquad \begin{aligned} &\text{(a)}\ \ \phi(x) \text{ is reproducible of order } k \\ &\text{(b)}\ \ \bar{\lambda} \approx 0, \end{aligned}$$

where $\bar{\lambda}$ is the largest eigenvalue of the matrix A given in (3.7). The translation invariant particle shape functions $\phi_i^h(x)$, corresponding to such basic shape function $\phi(x)$, will also be referred to as almost quasi-reproducing of order $(k+1)$.

We note that if the basic shape function $\phi(x)$ is reproducing of order k and satisfies

$$\xi_\alpha(x) \equiv x^\alpha - \sum_{i\in\mathbb{Z}^n} i^\alpha \phi(x-i) \approx C_\alpha, \quad \text{for all } |\alpha| = k+1,$$

where C_α is constant, then the elements of the matrix A are small (see Remark 3.2), and therefore $\bar{\lambda}$ is also small, implying that the basic shape function $\phi(x)$ is almost quasi-reproducible of order $(k+1)$. In 1-d, $\bar{\lambda} = |\xi_{k+1}|_1^2/(k+1)!^2$ and therefore a basic shape function $\phi(x)$ is almost quasi-reproducing of order $(k+1)$ if $\xi_{k+1}(x) \approx C$, where C is a constant.

Now, for a smooth function u defined on Ω, we have from Remark 3.11 that

$$\text{(5.8)} \qquad \|u - \tilde{\mathcal{I}}_h u\|_{1,\Omega} \le \sqrt{\bar{\lambda}}\, h^k |u|_{k+1,\Omega} + o(1)h^k,$$

where the shape functions are reproducing of order k. This is also true when $\|u - \tilde{\mathcal{I}}_h u\|_{1,\Omega}$ in the left-hand side of (5.8) is replaced by $|u - \tilde{\mathcal{I}}_h u|_{1,\Omega}$. For shape functions that are almost quasi-reproducing of order $(k+1)$, $\bar{\lambda}$ is small and it is clear from (5.8) that $\|u - \tilde{\mathcal{I}}_h u\|_{1,\Omega}$ will exhibit a higher order of convergence than the expected order, which is $O(h^k)$, in the pre-asymptotic range. Depending on the size of $\bar{\lambda}$, this pre-asymptotic range can be so large that for practical computation, the asymptotic range may not be visible. The same phenomenon is also true for $|u - \tilde{\mathcal{I}}_h u|_{1,\Omega}$.

We will illustrate this phenomenon with a numerical experiment. We consider the one-dimensional translation invariant RKP shape functions, reproducing of order $k = 1$, corresponding to the conical weight function given in (2.15) with $l = 10$ and $R = 3.40177$. In this case, the computed value of $|\xi_2|_1 \approx 2.04 * 10^{-6}$. Clearly, $|\xi_2|_1$ is small and these shape functions are almost quasi-reproducible of order 2. In Figures 5.2(a) and (b), we plotted loglog graphs of $E_0^h = \|u - \tilde{\mathcal{I}}_h u\|_{0,\Omega}$ and $E_1^h = |u - \tilde{\mathcal{I}}_h u|_{1,\Omega}$ with respect to $n = 1/h$ for $n = 20, 30, \ldots, 100$. Here $u(x) = x^4$, $\Omega = (0, 1)$, and $\tilde{\mathcal{I}}_h u(x)$ is the interpolant of $u(x)$ with respect to these shape functions.

We observe from Figure 5.2(b) that $|u - \tilde{\mathcal{I}}_h u|_{1,\Omega} \approx O(h^2)$, which is higher than $O(h)$, the order predicted by (5.1), with $l = 1$, and with the norm of the error replaced by the semi-norm. This shows that $|u - \tilde{\mathcal{I}}_h u|_{1,\Omega}$ may decrease at a higher order than predicted by (5.1), with $l = 1$, in the pre-asymptotic range, when almost quasi-reproducible shape functions of order $(k+1)$ are used in the interpolation. In fact, from (5.8), we see that we have to take $h << \sqrt{\bar{\lambda}} = |\xi_2|_1/2 = 1.02 * 10^{-6}$ to get the expected $|u - \tilde{\mathcal{I}}_h u|_{1,\Omega} \approx O(h)$ in our experiment (*i.e.,* in Figure 5.2(b)). However, we note from Figure 5.2(a) that $\|u - \tilde{\mathcal{I}}_h u\|_{0,\Omega} \approx O(h^2)$, which is same as predicted by (5.1) with $l = 0$.

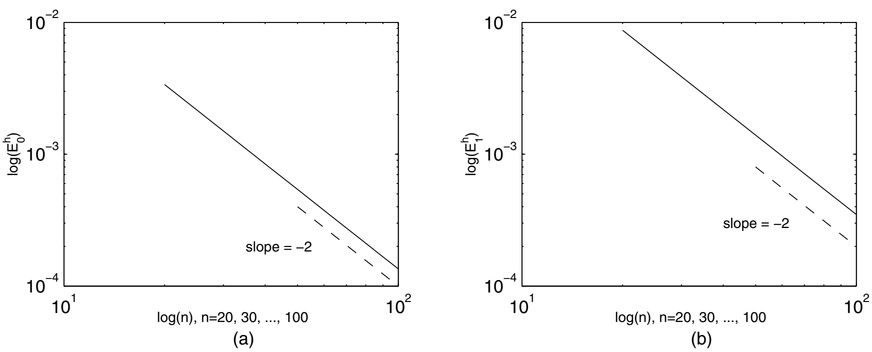

Fig. 5.2 (a), (b). The loglog graph of $E_0^h = \|u - \tilde{\mathcal{I}}_h u\|_{0,\Omega}$ and $E_1^h = |u - \tilde{\mathcal{I}}_h u|_{1,\Omega}$ with respect to $n = 1/h$, where $u(x) = x^4$ and $\tilde{\mathcal{I}}_h u$ is the interpolant of $u(x)$ using shape functions that are reproducing of order 1 and almost quasi-reproducing of order 2

Nevertheless, we reiterate that we get the expected order of convergence for $|u - \tilde{\mathcal{I}}_h u|_{1,\Omega}$, as obtained from (5.2), when the shape functions are reproducing of order k and are neither quasi-reproducing nor almost quasi-reproducing of order $(k+1)$. A loglog plot of the values of $|u - \tilde{\mathcal{I}}_h u|_{1,\Omega}$ in Table 4.1 with respect to $n = 1/h$ shows that the order of convergence is $O(h)$ when h is small; we have not included this plot.

6 Comparison of translation invariant and standard RKP-shape functions

In the last section, we discussed the interpolation error and the approximation error for particle shape functions. We also observed that the values in Table 4.3 are consistently larger than the values in Table 4.1. Recall that Tables 4.1 and 4.3 provide the values of $|u - \tilde{\mathcal{I}}_h u|_{1,\Omega}$, where $\tilde{\mathcal{I}}_h u$ is the interpolant of $u = x^4$ on $\Omega = (0, 1)$ corresponding to translation invariant and standard RKP shape functions, respectively. It is interesting to compare the errors (not the norm of the errors) generated by using these two classes of shape functions. In this section, we numerically study the errors $u - \tilde{\mathcal{I}}_h u$ and $u' - (\tilde{\mathcal{I}}_h u)'$, where u is a given smooth function and the interpolant $\tilde{\mathcal{I}}_h u$ is obtained using translation invariant as well as standard RKP shape functions. We will also compare the errors $u' - (P_h^1 u)'$ and $u' - (P_h^2 u)'$, where $P_h^1 u$, $P_h^2 u$ are H^1-projections of u onto the spaces spanned by these two classes of shape functions.

Throughout this section, we will consider $u(x) = (x - .5)^2$, $x \in \Omega = (0, 1)$, and will use the RKP shape functions, translation invariant as well as standard, that are reproducing of order $k = 1$, corresponding to the conical weight function given in (2.15) with $l = 2$ and $R = 2.5$. In this case, $|\xi_2|_1 = 0.1137$, and the corresponding translation invariant RKP shape functions are not almost quasi-optimal of order 2. We have considered a different function u in this section to illustrate (3.5) in Lemma 3.1. Also this u is symmetric about $x = 0.5$, so we should expect symmetry in the error.

We first present graphs of $(u - \tilde{\mathcal{I}}_h u)(x), 0 \le x \le 1$ for two different values of $h = 1/n$ in Figures 6.1(a)–(d).

We first remark that $(u - \tilde{\mathcal{I}}_h u)(x) = \xi_2^h(x)$ (cf.(3.4)), since the translation invariant shape functions that we considered are reproducing order 1 and u is quadratic. In Figures 6.1(a) and (c), we have plotted the graphs of $(u - \tilde{\mathcal{I}}_h u)(x)$ for $n = 1/h = 20, 40$, respectively, where the interpolant $\tilde{\mathcal{I}}_h u$ corresponds to the translation invariant RKP shape functions. It is clear from these two figures that $(u - \tilde{\mathcal{I}}_h u)(x) = \xi_2^h(x)$ is periodic throughout the interval $\Omega = (0, 1)$. This illustrates (3.5) in Lemma 3.1. Also the values of $(u - \tilde{\mathcal{I}}_h u)(x)$ are symmetric about $x = 0.5$ and their magnitude decrease as n increases from 20 to 40.

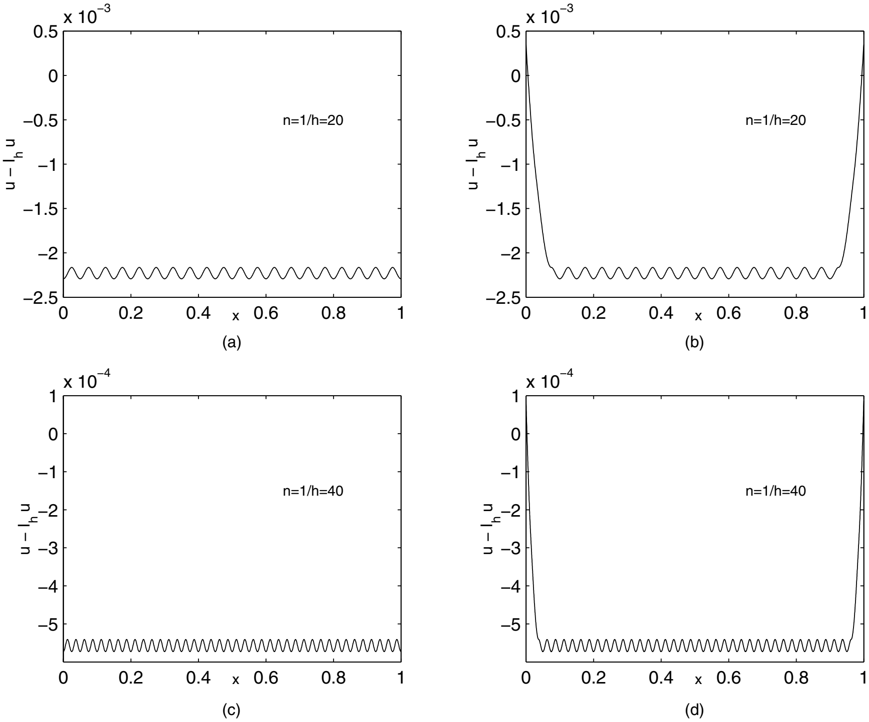

Fig. 6.1 (a)–(d). Figures 6.1(a) and (c) are graphs of $(u - \tilde{\mathcal{I}}_h u)(x)$, $0 \le x \le 1$, for $n = 1/h = 20, 40$, respectively. Here $u(x) = (x - .5)^2$ and $\tilde{\mathcal{I}}_h u$ is the interpolant of u using translation invariant RKP shape functions that are reproducing of order 1. Figures 6.1(b) and (d) are graphs of $(u - \tilde{\mathcal{I}}_h u)(x)$, $0 \le x \le 1$, for $n = 1/h = 20, 40$, respectively, where $\tilde{\mathcal{I}}_h u$ is the interpolant of u using standard RKP shape functions that are reproducing of order 1

In Figures 6.1(b) and (d), we have plotted the graphs of $(u - \tilde{\mathcal{I}}_h u)(x)$ for $n = 1/h = 20, 40$, respectively, when the interpolant $\tilde{\mathcal{I}}_h u$ corresponds to the standard RKP shape functions. Also in this case, the values of $(u - \tilde{\mathcal{I}}_h u)(x)$ are symmetric about $x = 0.5$, and their magnitude decrease as n increases from 20 to 40. Moreover, the behavior and magnitude of $(u - \tilde{\mathcal{I}}_h u)(x)$ in Figures 6.1(b) and (d) in the middle part of the domain $\Omega = (0, 1)$, *e.g.* in (.4,.6), is similar to that in Figures 6.1(a) and (c), respectively. This is the result of the translation invariance of standard RKP particle shape functions, corresponding to the uniformly distributed particles in the interior of the domain $\Omega = (0, 1)$, sufficiently away from the boundary. But, in contrast to Figures 6.1(a) and (c), Figures 6.1(b) and (d) show a boundary layer in $(u - \tilde{\mathcal{I}}_h u)(x)$. We computed the ratio:

$$L = \frac{A - m}{B - m} = \left(A - \frac{B + C}{2} \right) \Big/ \left(\frac{B - C}{2} \right), \tag{6.1}$$

where

$$A = (u - \tilde{\mathcal{I}}_h u)(1), \qquad B = \max_{.4 \le x \le .6} (u - \tilde{\mathcal{I}}_h u)(x),$$

$$C = \min_{.4 \le x \le .6} (u - \tilde{\mathcal{I}}_h u)(x), \qquad m = \frac{B + C}{2}.$$

In other words, the quantity L measures the factor by which the maximum deviation of error at the boundary from m — which is the mean of the error near the middle of the domain — "shoots off" with respect to the maximum deviation of the error in the middle of the domain from m. We note that L is almost independent h for small h, since the numerator and denominator of L are of the same order in h. We obtained $L \approx 40$, and as expected, this value was about same for $n = 20$ and 40.

In Figures 6.2(a)–(d), we present the graphs of $u' - (\tilde{\mathcal{I}}_h u)'$ for the same values of n.

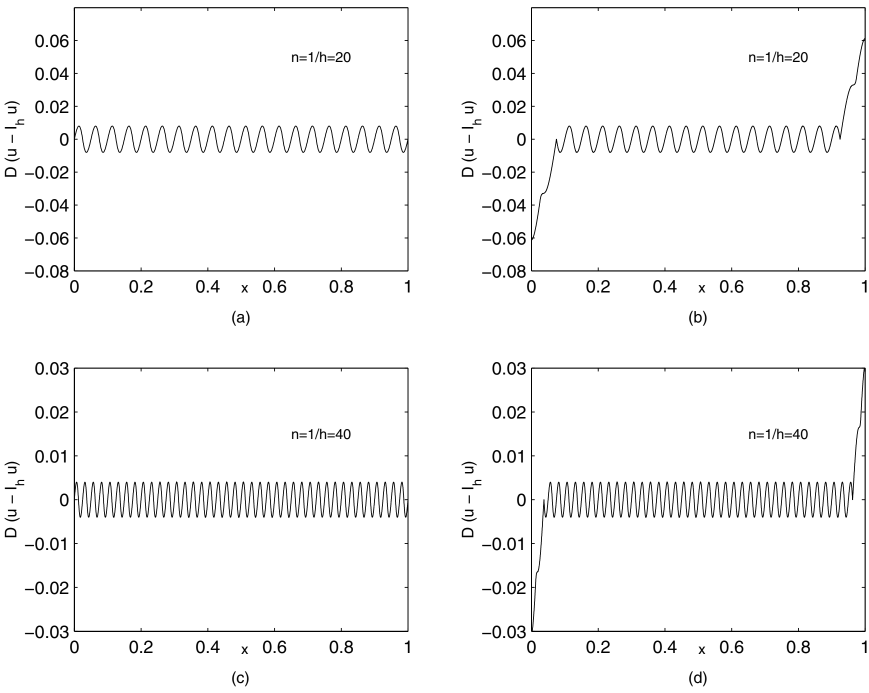

Fig. 6.2 (a)–(d). Figures 6.2(a) and (c) are graphs of $(u' - (\tilde{\mathcal{I}}_h u)')(x)$, $0 \le x \le 1$, for $n = 1/h = 20, 40$, respectively. Here $u(x) = (x - .5)^2$ and $\tilde{\mathcal{I}}_h u$ is the interpolant of u using translation invariant RKP shape functions that are reproducing of order 1. Figures 6.2(b) and (d) are graphs of $(u' - (\tilde{\mathcal{I}}_h u)')(x)$, $0 \le x \le 1$, for $n = 1/h = 20, 40$, respectively, where $\tilde{\mathcal{I}}_h u$ is the interpolant of u using standard RKP shape functions that are reproducing of order 1

As before, we have plotted the graphs of $(u' - (\tilde{\mathcal{I}}_h u)')(x)$ for $n = 1/h = 20, 40$ in Figures 6.1(a) and (c), respectively, where the interpolant $\tilde{\mathcal{I}}_h u$ corresponds to the translation invariant RKP shape functions. Clearly, $(u' - (\tilde{\mathcal{I}}_h u)')(x)$ is periodic throughout the interval (0, 1), as expected, being the derivative of a periodic function. Also the values of $(u' - (\tilde{\mathcal{I}}_h u)')(x)$ are anti-symmetric about $x = 0.5$, and their magnitude decrease as $n = 1/h$ increases.

Also as before, we have plotted the graphs of $(u' - (\tilde{\mathcal{I}}_h u)')(x)$ for $n = 1/h = 20, 40$ in Figures 6.2(b) and (d), respectively, where the interpolant $\tilde{\mathcal{I}}_h u$ corresponds to the standard RKP shape functions. The values of $(u' - (\tilde{\mathcal{I}}_h u)')(x)$ decreases as n increases from 20 to 40 as expected. Also the magnitude of the $(u' - (\tilde{\mathcal{I}}_h u)')(x)$, for fixed n, increases near the endpoints of the interval (0,1), which corresponds to the boundary layers in Figures 6.1(b) and (d). We computed the ratio L as in (6.1) (with $A = (u' - (\tilde{\mathcal{I}}_h u)')(1)$, and with similar changes in B and C), and we obtained $L \approx 6.6$, which did not change with n.

We saw that interpolation error gives rise to boundary layer when standard RKP shape functions are used. Now we will investigate whether similar phenomenon occurs in the approximation error. In Figures 6.3(a)–(d), we have plotted $u' - (P_h^1 u)'$ and $u' - (P_h^2 u)'$ for two different values of $h = 1/n$, where $u = (x - 0.5)^2,\ x \in (0, 1)$. Here, $P_h^1 u$ is the H^1-projection of u onto space spanned by translation invariant RKP shape functions, defined as

$$(6.2)\qquad \begin{aligned} &P_h^1 u \in \operatorname{span}\{\phi_i^h\}_{i \in A^h}, \\ &\int_\Omega [u' - (P_h^1 u)'](\phi_i^h)'\, dx = 0, \quad i \in A_h, \end{aligned}$$

where $A^h = \{m \in \mathbb{Z} : \mathring{\eta}_m^h \cap \Omega \neq \emptyset\}$, as previously defined. $P_h^2 u$ is the H^1-projection of u onto space spanned by standard RKP shape functions, defined as

$$(6.3)\qquad \begin{aligned} &P_h^2 u \in \operatorname{span}\{\phi_i^h\}_{i \in A_\Omega}, \\ &\int_\Omega [u' - (P_h^2 u)'](\phi_i^h)'\, dx = 0, \quad i \in A_\Omega, \end{aligned}$$

where $A_\Omega = \{j \in \mathbb{Z} : x_j^h \in \bar{\Omega}\}$.

In Figures 6.3(a) and (c), we plotted $(u' - (P_h^1 u)')(x)$ for $n = 40, 60$ respectively. We clearly see that the values of $(u' - (P_h^1 u)')(x)$ are anti-symmetric about $x = 0.5$, and their magnitude decrease as n increases. We also see that $(u' - (P_h^1 u)')(x)$ is periodic in the interior of the interval (0, 1), in contrast to Figures 6.2(a) and (c) where the error is periodic throughout the interval. But the graphs in Figures 6.3(a) and (c) do not show any substantial change in the behavior of the error near the boundary.

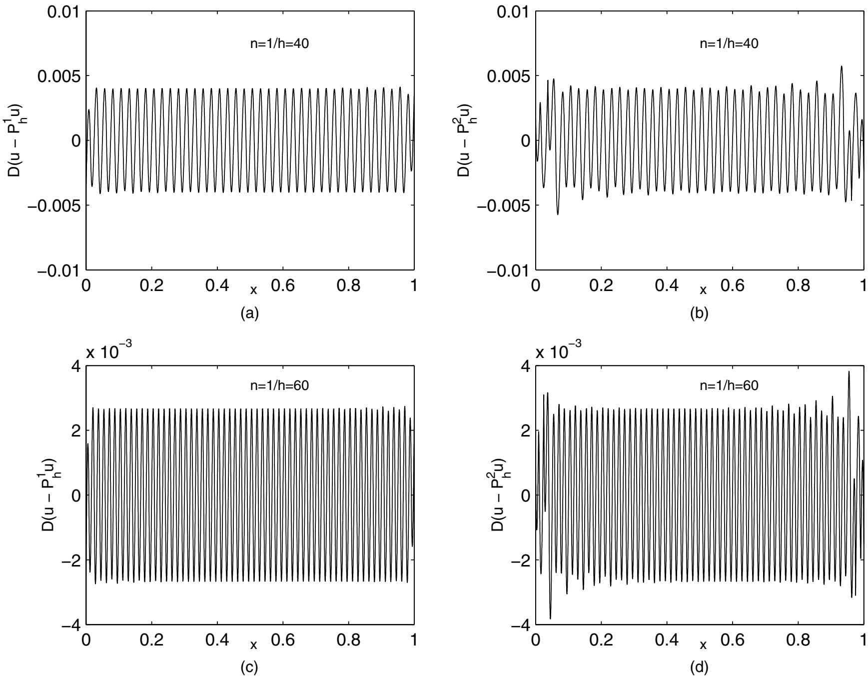

Fig. 6.3 (a)–(d). Figures 6.3(a) and (c) are graphs of $(u' - (P_h^1 u)')(x)$, $0 \leq x \leq 1$, for $n = 1/h = 20, 40$, respectively. Here $u(x) = (x - .5)^2$ and $P_h^1 u$ is the H^1-projection of u onto the space spanned by translation invariant RKP shape functions that are reproducing of order 1. Figures 6.3(b) and (d) are graphs of $(u' - (P_h^2 u)')(x)$, $0 \leq x \leq 1$, for $n = 1/h = 20, 40$, respectively, where $P_h^2 u$ is the H^1-projection of u onto the space spanned by standard RKP shape functions that are reproducing of order 1

Figures 6.3(b) and (d) are plots of $(u' - (P_h^2 u)')(x)$ for $n = 40, 60$ respectively. The values of $(u' - (P_h^2 u)')(x)$ are also anti-symmetric about $x = 0.5$, and their magnitude decrease as n increases. Though we do not see any strong boundary layer in Figures 6.3(b) and (d), as observed in Figures 6.2(b) and (d), we certainly see that $(u' - (P_h^2 u)')(x)$ shoots off for certain values of x close to the boundary, and the magnitude of these "shoot-offs" are higher nearer the boundary. We further observe from Figure 6.3(b) that $(u' - (P_h^2 u)')(x)$ does not show any periodicity in the interior of (0, 1) for $n = 40$, as in Figure 6.3(a). But for $n = 60$, the behavior and magnitude of $(u' - (P_h^2 u)')(x)$ in the interior of the interval (0, 1), *e.g.,* in (4.5, 5.5) in Figure 6.3(d) is similar to $(u' - (P_h^1 u)')(x)$, in Figure 6.3(c). In particular, $(u' - (P_h^2 u)')(x)$ is periodic in (4.5, 5.5). A comparison of all the four plots in Figure 6.3 suggests that, as n increases, the behavior and magnitude of $(u' - (P_h^2 u)')(x)$ in the interior of the interval (0, 1) is similar to that of $(u' - (P_h^1 u)')(x)$. This was also supported by our computations with higher values of n, which we did not include here. Computing with higher values of

n also suggested that the size of the interior region, where $(u' - (P_h^2 u)')(x)$ is periodic, increases as n is increased.

Thus, Figures 6.1–6.3 suggest that interpolation or approximation of smooth functions by standard RKP shape function may have "erratic" behavior near the boundary of the domain, and in particular may not be accurate near the boundary. We believe that this behavior is due to the fact that standard RKP shape functions use particles only inside the domain. On the other hand, interpolation or approximation of smooth functions by translation invariant shape functions, which use particles also outside the domain, does not show this behavior.

7 Conclusion

We have addressed the issue of the selection of translation invariant particle shape functions that yields efficient approximation. Our main conclusions and observations are:

- From a given collection of basic shape functions that are reproducing of order k, we choose the basic shape function with least value of $\bar{\lambda}$, where $\bar{\lambda}$ is the largest eigenvalue of the matrix A given in (3.7). The translation invariant particle shape functions corresponding to this shape function will yield efficient approximation among the particle shape functions corresponding to other basic shape function in the collection. This selection process is based on Theorem 3.1. The same process also appear to be successful for standard RKP shape functions.
- The notion of quasi-reproducing shape function of order k in the approximation analysis, as opposed to the notion of reproducing shape functions of order k, gives correct order of convergence for the approximation error.
- The interpolation error may decrease at a rate higher, in the pre-asymptotic region, than is predicted by the theory. This pre-asymptotic rate could be so large that it is not visible for practical accuracy.
- Standard RKP shape functions gives rise to boundary layers in the inter polation error. The projection error, corresponding to these shape functions, also shows a similar irregular boundary behavior. The translation invariant RKP shape functions do not show this behavior.

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