Superconvergence in the Generalized Finite Element Method

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Abstract

In this paper, we address the problem of the existence of superconvergence points of approximate solutions, obtained from the Generalized Finite Element Method (GFEM), of a Neumann elliptic boundary value problem. GFEM is a Galerkin method that uses non-polynomial shape functions, and was developed in [4, 5, 24]. In particular, we show that the superconvergence points for the gradient of the approximate are zeros of certain systems of non-linear equations that do not depend on the solution of the boundary value problem. For approximate solutions with second derivatives, we have also characterized the superconvergence points of the second derivatives of the approximate solution as the roots of certain systems of non-linear equations. We note that it is easy to construct smooth generalized finite element approximation.

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1 Introduction

The superconvergence in the finite element method (FEM) is a phenomenon, where the order of convergence of the finite element error, at certain special points in an element, is higher than the order of convergence of the maximum of the finite element error over that element. These special points are called natural superconvergence points. To the best of our knowledge, this phenomenon was

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first addressed in [26], and the term superconvergence was first used in [17]. Superconvergence has been extensively studied ([2, 11, 18, 22, 27, 28, 33, 34, 36, 37] to name a few) and there are more than 1000 papers available on the subject. An extensive bibliography (before 1998) on superconvergence is available in [20], where as many references on 3-dimensional problems can be found in [19]. Moreover, there have been several books written on superconvergence in the context of the finite element method, *e.g.*, [1, 6, 13, 14, 23, 35, 38].

Typically, superconvergence in the FEM has been studied for triangular meshes, as well as for quadrilateral meshes with tensor-product elements, but rarely for serendipity elements. Moreover, the mesh is required to have some local regularity, e.g., elements are essentially translation invariant. Also, most of these studies are confined within the interior of the underlying domain, and only a few address the issue of superconvergence up to the boundary.

Later, a systematic approach was introduced in the analysis of superconvergence in [9, 10], in the context of the finite element method. This analysis allows more general meshes, where the elements could be grouped into translation invariant "cells" (in contrast to elements being translation invariant). The cells could contain arbitrary number of elements of different types. It was shown in these studies that the existence of natural superconvergence points was equivalent to the existence of roots of a system of polynomial equations. Moreover, the superconvergence points are obtained from these roots, which (the roots) are computed numerically. In special situations, the system of equations can be written explicitly and roots can be computed analytically, as shown in [36, 37]. In the context of finite element approximations of solutions of the Poisson's equation and the Laplace equation, superconvergence was studied in [10] for four different types of triangular meshes, as shown in Figure 1.1, as well as for square mesh with tensor-product, intermediate, and serendipity elements.

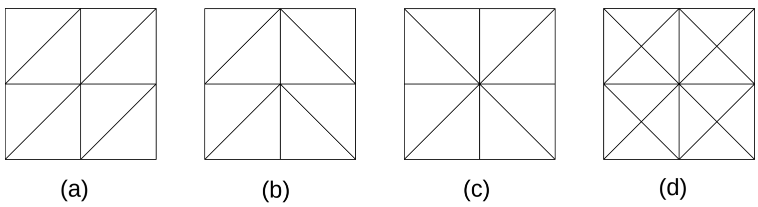


Figure 1.1: (a) Regular pattern; (b) Chevron pattern; (c) Union Jack pattern; (d) Cris-cross pattern.

We state some of these results (from [9, 10]; see also page 354 of [6]) in the context of approximation of the solution of the Poisson equation. It was shown that for triangular meshes, there are no natural superconvergence points for a mesh (a) with regular pattern when $p > 2$ and even; (b) with Chevron pattern when p is even; (c) with Union Jack pattern for any p. Also, for a square mesh with serendipity elements, it was shown that (a) there are 4 natural superconvergence points and a superconvergence line for $p = 3$; (b) there are no natural superconvergence points when $p \geq 4$ and is even, where as, there are 3 such points when $p \geq 5$ and is odd. The coordinates of all these points can be found in [10]. These results illustrate the intrinsic complexity of the superconvergence phenomenon, and they indicate that the approach is quite general to analyze this complexity. In [8], the study of superconvergence was extended to the mesh cells near the boundary of the domain. This approach was also used in [7, 8] to study the effectiveness of various a-posteriori error estimators.

In this paper, we will address the problem of superconvergence in the context of Generalized Finite Element Method (GFEM). This method was introduced in [4] and later developed and elaborated in [5, 24]. It is a Galerkin method that uses a mesh only minimally, and allows the use of non-polynomial shape functions. We will follow the approach of [10] in the analysis of superconvergence presented in this paper. We will address the superconvergence in GFEM only in the interior of the underlying domain.

The main results of this paper are Theorems 4.1 and 4.2 given in Section 4. Theorem 4.1 shows that the superconvergence points can be obtained by finding the zeros of a system of equations that does not depend on the exact solution of the boundary value problem. Theorem 4.2 addresses the superconvergence points for the second derivatives of the generalized finite element error. We mention that GFEM allows smooth approximation, in particular a C^2 approximation, which in turn allows us to address the superconvergence of second derivatives of the error.

We briefly describe the organization of this paper. In Section 2, we describe the GFEM and review the main approximation result. In Section 3, we discuss the so called interior estimates, which is crucial for the superconvergence analysis. In Section 4, we present the main results of this paper, namely, Theorems 4.1 and 4.2. We present an example in Section 5 that illuminates the results obtained in Section 4.

2 Generalized Finite Element Methods

In this section, we briefly describe the GFEM in the context of the approximation of the solution of a linear Neumann boundary value problem.

Let $\Omega \subset \mathbb{R}^2$ be a domain with piecewise smooth boundary $\partial\Omega$. We consider the Neumann problem

$$\begin{cases} \triangle u = f, \text{ on } \Omega, \\ \dfrac{\partial u}{\partial n} = g, \text{ on } \partial\Omega, \end{cases} \tag{2.1}$$

where

$$\int_\Omega f\,dx + \int_{\partial\Omega} g\,ds = 0. \tag{2.2}$$

We now give the standard variational formulation of the above problem. Let

$$B(u,v) = \int_\Omega \left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right] dx\,dy$$

and

$$F(v) = \int_\Omega f v \, dx + \int_{\partial\Omega} g v \, ds,$$

where we assume $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$. We will often use the notation

$$B_M(u, v) = \int_M \left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right] dx \, dy, \tag{2.3}$$

where $M \subset \Omega$. The weak formulation of (2.1) reads,

$$\begin{cases} \text{Find } u \in H^1(\Omega) \text{ satisfying} \\ B(u, v) = F(v) \text{ for all } v \in H^1(\Omega). \end{cases} \tag{2.4}$$

The above problem is uniquely solvable up to a constant; we assume

$$\int_\Omega u \, dx = 0,$$

which ensures a unique solution of (2.4).

The GFEM to approximate the solution of (2.4) is a Galerkin method where the construction of trial and test spaces depend on a (i) *partition of unity* (PU), and (ii) *local approximating spaces.*

(i) For $0 < h \leq 1$, a parameter, let $\{\omega_j^h\}_{j=1}^N$ be convex sub-domains of Ω with $N = N(h)$ such that $d_j^h \equiv \text{diam}(\omega_j^h) \leq 2h$ for $j = 1, 2, \ldots, N$. We assume that for each value of h,

$$\bigcup_{j=1}^{N(h)} \omega_j^h = \Omega, \tag{2.5}$$

and that any $x \in \Omega$ belongs to at most κ of the sets ω_j^h, where κ is independent of h. The sub-domains ω_j^h are called *patches.* Clearly, $\{\omega_j^h\}_{j=1}^N$ is an open cover of Ω. Let $\{\phi_j^h\}_{j=1}^N$ be a family of C^2 functions defined on Ω satisfying

$$\phi_j^h(x, y) = 0, \qquad \text{for } (x, y) \in \Omega \setminus \omega_j^h, \ 1 \leq j \leq N(h), \tag{2.6}$$

$$\sum_{j=1}^{N(h)} \phi_j^h(x, y) = 1, \qquad \text{for } (x, y) \in \Omega, \tag{2.7}$$

$$\max_{(x,y)\in\Omega} |\phi_j^h(x, y)| \leq C_1, \qquad \text{for } 1 \leq j \leq N(h), \text{ and} \tag{2.8}$$

$$\max_{(x,y)\in\Omega} |D^\alpha \phi_j^h(x, y)| \leq \frac{C_2}{|\, d_j^h|^{|\alpha|}}, \qquad \text{for } |\alpha| \leq 2 \text{ and } 1 \leq j \leq N(h) \tag{2.9}$$

where α is a multi-index and constants C_1, C_2 are independent of h. It is clear from (2.7) that $\{\phi_j^h\}_{j=1}^N$ form a *partition of unity.*

(ii) To each patch ω_j^h, we associate an m_j-dimensional space V_j^h of functions, defined on $\bar{\omega}_j^h$, given by

$$V_j^h = \left\{ \xi_j^h : \xi_j^h = \sum_{i=1}^{m_j} b_{ij}^h \xi_{ji}^h, \ b_{ij}^h \in \mathbb{R}, \ \xi_{ji}^h \in H^1(\omega_j^h) \cap C(\bar{\omega}_j^h) \right\}, \tag{2.10}$$

and we assume that V_j^h contains constant functions. The space V_j^h is called a *local approximating space*. In the rest of this paper, we will suppress the h in $\omega_j^h, d_j^h, N(h), \phi_j^h, V_j^h, \xi_j^h$, and ξ_{ji}^h, and refer to them as $\omega_j, d_j, \phi_j, N, V_j, \xi_j$, and ξ_{ji} respectively for notational clarity and convenience, with the understanding that they depend on h.

The trial and test spaces in GFEM is given by

$$\begin{aligned} S^{GFEM} &= \left\{ \psi = \sum_{j=1}^{N} \phi_j \xi_j;\ \text{where } \xi_j \in V_j \right\} \\ &= \text{span}\left\{ \eta_{ji} = \phi_j \xi_{ji}; 1 \le i \le m_j \text{ and } 1 \le j \le N \right\}. \end{aligned} \tag{2.11}$$

The functions $\{\eta_{ji}\}$ are the shape functions of S^{GFEM}. Finally, the GFEM to approximate the solution of (2.4) is given by

$$\begin{cases} \text{Find } u_{GFEM} \in S^{GFEM} \text{ satisfying} \\ \int_\Omega u_{GFEM} = 0, \\ B(u_{GFEM}, v) = F(v) \text{ for all } v \in S^{GFEM}. \end{cases} \tag{2.12}$$

This problem has a unique solution and is equivalent to a system of linear algebraic equations. Specifically, if we write

$$u_{GFEM} = \sum_{j=1}^{N} \sum_{i=1}^{m_j} c_{ji} \eta_{ji},$$

then (2.12) yields the linear system

$$\begin{cases} \displaystyle\sum_{j=1}^{N} \sum_{i=1}^{m_j} c_{ji} \int_{\omega_j} \eta_{ji} = 0 \\ \displaystyle\sum_{j=1}^{N} \sum_{i=1}^{m_j} B(\eta_{lk}, \eta_{ji}) c_{ji} = F(\eta_{lk}), 1 \le k \le m_l, 1 \le l \le N. \end{cases} \tag{2.13}$$

We note that the shape functions $\{\eta_{ji}\}$ could be linearly dependent, and thus the dimension of the null space of the matrix in (2.13) could be greater than zero. In this case, the system (2.13) does not have a unique solution. However, u_{GFEM} is unique, *i.e.*, if $\{c_{ji}^{(1)}\}$ and $\{c_{ji}^{(2)}\}$ are two solutions of (2.13), then

$$u_{GFEM} = \sum_{j=1}^{N} \sum_{i=1}^{m_j} c_{ji}^{(1)} \eta_{ji} = \sum_{j=1}^{N} \sum_{i=1}^{m_j} c_{ji}^{(2)} \eta_{ji}.$$

For examples of linearly dependent shape functions in GFEM, see [3]. In this paper, we will use a particular S^{GFEM}, which will have linearly independent shape functions.

We now present two results on approximation properties of S^{GFEM}, which in turn give an error estimate for $\|u - u_{GFEM}\|_{H^1(\Omega)}$. Suppose $u \in H^1(\Omega)$ can be accurately approximated on ω_j by a function $\xi_j^u \in V_j$; specifically, suppose

$$\|u - \xi_j^u\|_{L_2(\omega_j)}^2 \le \epsilon_1^2(j)$$

and

$$|u - \xi_j^u|_{H^1(\omega_j)}^2 \le \epsilon_2^2(j).$$

Define

$$\xi^u = \sum_{j=1}^N \phi_j \xi_j^u \in S^{GFEM}$$

Then we have the following two results ([5, 24]).

Theorem 2.1 *Suppose $u \in H^1(\Omega)$. Then*

$$\|u - \xi^u\|_{L_2(\Omega)} \le C_1 \kappa^{1/2} \left(\sum_{j=1}^N \epsilon_1^2(j) \right)^{1/2}$$

and

$$|u - \xi^u|_{H^1(\Omega)} \le (2\kappa)^{1/2} \left(C_2^2 \sum_{j=1}^N \frac{\epsilon_1^2(j)}{d_j^2} + C_1^2 \sum_{j=1}^N \epsilon_2^2(j) \right)^{1/2},$$

where C_1, C_2 are as in (2.8), (2.9), respectively, and $d_j \equiv diam(\omega_j)$. $\square$

Remark 2.1 We note that the above theorem is true even when the patches ω_j are non-convex.

Theorem 2.2 *Suppose $u \in H^1(\Omega)$. Suppose the patches ω_j satisfy the following assumption:*

For all $1 \le j \le N$, there exists $C_3 > 0$, independent of j, such that

$$\|v\|_{L_2(\omega_j)} \le C_3 d_j \, |v|_{H^1(\omega_j)}, \quad \text{for all } v \in H^1(\omega_j) \text{ satisfying } \int_{\omega_j} v \, dx = 0, \quad (2.14)$$

where $d_j \equiv diam(\omega_j)$.

Then, there exists $\tilde{\xi}_j^u \in V_j$ such that

$$\tilde{\xi}^u = \sum_{j=1}^N \phi_j \tilde{\xi}_j^u \in S^{GFEM}$$

satisfies

$$\|u - \tilde{\xi}^u\|_{L_2(\Omega)} \le C_5 \left(\sum_{j=1}^N d_j^2 \, \epsilon_2^2(j) \right)^{1/2}$$

and

$$|u - \tilde{\xi}^u|_{H^1(\Omega)} \leq C_6 \left(\sum_{j=1}^{N} \epsilon_2^2(j) \right)^{1/2},$$

where C_5, C_6 *depend on* C_1, C_2, C_3.□

Remark 2.2 It is shown in [3] that (2.14) holds when the patches ω_j are convex. The precise dependence of C_5, C_6 on C_1, C_2, C_3, and the dependence of the Poincaré constant C_3 on the geometric data of ω_j is also given in [3]. In the rest of the paper, we will not differentiate between various constants, and instead will use a generic constant C.

Theorem 2.2 gives an error estimate for the GFEM. Suppose the hypothesis (2.14) is satisfied and suppose u is the solution of (2.4). Then from Theorem 2.2 we have

$$\|u - u_{GFEM}\|_{H^1(\Omega)} \leq \|u - \tilde{\xi}^u\|_{H^1(\Omega)} \leq C \left(\sum_{j=1}^{N} \epsilon_2^2(j) \right)^{1/2}.$$

It will be useful to state this estimate in the form

$$\|u - u_{GFEM}\|_{H^1(\Omega)} \leq C \left(\sum_{j=1}^{N} \inf_{\xi_j \in V_j} \|u - \xi_j\|^2_{H^1(\omega_j)} \right)^{1/2}. \qquad (2.15)$$

To obtain the main result of this paper, we will impose additional restrictions on the patches $\{\omega_j\}$, the partition of unity $\{\phi_j\}$, and the local approximation spaces V_j. We list them as three assumptions.

Assumption A In addition to (2.5), we assume that

$$\omega_j^* \subset \omega_j, \quad 1 \leq j \leq N,$$

where ω_j^* is a ball of diameter d_j^*, and there exists $0 < \sigma < 1$, independent of the parameter h, such that

$$d_j^* \geq \sigma d_j, \quad 1 \leq j \leq N. \qquad (2.16)$$

Assumption B In addition to (2.6)–(2.9), we assume that

$$\phi_j(x, y) = 1, \quad \text{for } (x, y) \in \omega_j^*.$$

Since $\{\phi_j\}$ is a partition of unity, it is clear that $\omega_j^* \cap \omega_i^* = \emptyset$ for $j \neq i$, and $\phi_j(x, y) = 0$ for $(x, y) \in \omega_i^*$ when $i \neq j$.

Assumption C We consider $V_j = \mathcal{P}^k(\omega_j)$, $1 \leq j \leq N$, where $\mathcal{P}^k(\omega_j)$ is the space of polynomials of degree k on ω_j. We assume that for $1 \leq p \leq \infty$,

$$\text{(a)} \quad \|\xi\|_{W_p^s(\omega_j)} \leq C\|\xi\|_{W_p^s(\omega_j^*)}, \text{ for all } \xi \in V_j \text{ and } 0 \leq s \leq k, \qquad (2.17)$$

where C depends on k, but not on j or h;

(b) $\|\xi\|_{W_p^t(\omega_j)} \leq C d_j^{-(t-s)} \|\xi\|_{W_p^s(\omega_j)}$, for all $\xi \in V_j$ and $0 \leq s \leq t \leq k$. (2.18)

where C depends on s and t, but is independent of j and h.

Furthermore, when $\bar{\omega}_j \cap \partial\Omega \neq \emptyset$, we may allow ω_j^* to be a portion of a ball $b_j \not\subset \Omega$ with center inside ω_j, such that $\omega_j^* \equiv b_j \cap \Omega$ satisfies $\omega_j^* \subset \omega_j$ and

$$0 < B_1 \leq \frac{|\omega_j^*|}{|b_j|} \leq B_2 < 1, \tag{2.19}$$

where the constants B_1, B_2 are independent of j and h, and $|\omega_j^*|$, $|b_j|$ are the areas of the sets ω_j^*, b_j respectively.

We note that (2.17) holds if the patches ω_j satisfy some reasonable assumptions. For example, let B_j be the smallest ball containing ω_j with the same center as ω_j^*. If

$$1 \leq \frac{diam(B_j)}{diam(\omega_j^*)} \leq C, \tag{2.20}$$

then one can show that (2.17) is satisfied.

We will often denote S^{GFEM} by

$$S^h = S^{h,k}, \tag{2.21}$$

where k indicates degree of the polynomials used in the V_j's. Let $\{\xi_{ji}\}_{i=1}^{m_j}$ be a basis of V_j for $j = 1, 2, \ldots, N$. Then

$$\eta_{ji} = \phi_j \xi_{ji}, \quad 1 \leq i \leq m_j, \ 1 \leq j \leq N$$

are the shape functions of S^h. In the following proposition, we give an easy proof that $\{\eta_{ji}\}$ is a linearly independent set.

Proposition 2.1 *The set of shape functions $\{\eta_{ji}, 1 \leq i \leq m_j, 1 \leq j \leq N\}$ is linearly independent.*

Proof: Suppose the set is linearly dependent and there are constants $c_{ji}, 1 \leq i \leq m_j, 1 \leq j \leq N$, not all zero such that

$$\sum_{j=1}^{N} \sum_{i=1}^{m_j} c_{ji}\, \eta_{ji}(x,y) = 0, \quad \text{for all } (x,y) \in \Omega. \tag{2.22}$$

Without loss of generality, suppose the constant $c_{j_0,i_0} \neq 0$ for some $1 \leq i_0 \leq m_j, 1 \leq j_0 \leq N$. From Assumption B, we have $\phi_{j_0}(x,y) = 1$ for $(x,y) \in \omega_{j_0}^*$, and $\phi_j(x,y) = 0$ for $(x,y) \in \omega_{j_0}^*$, $j \neq j_0$. Therefore using (2.22), we get

$$\begin{aligned} \sum_{j=1}^{N} \sum_{i=1}^{m_j} c_{ji}\, \eta_{ji}(x,y) &= \sum_{i=1}^{m_{j_0}} c_{j_0,i} \phi_{j_0}(x,y) \xi_{j_0,i}(x,y) \\ &= \sum_{i=1}^{m_{j_0}} c_{j_0,i} \xi_{j_0,i}(x,y) = 0, \quad \text{for all } (x,y) \in \omega_{j_0}^*. \end{aligned}$$

But $\{\xi_{j_0 i}\}_{i=1}^{m_{j_0}}$ is a basis of V_{j_0}, and thus we conclude from above that $c_{j_0,i} = 0$ for $1 \le i \le m_{j_0}$. In particular, we have $c_{j_0,i_0} = 0$, which is a contradiction. Hence $\{\eta_{ji}, 1 \le i \le m_j, 1 \le j \le N\}$ is linearly independent. □

As mentioned before, the set of shape functions $\{\eta_{ji}\}$ for the GFEM may be linearly dependent. In contrast, the set of shape functions for the GFEM obtained using the patches ω_j and the partition of unity functions ϕ_j satisfying assumptions A and B is linearly independent. Thus the linear system (2.13) has a unique solution.

We now present a few examples of patches $\{\omega_j\}$ and partition of unity functions, defined relative to these patches, satisfying assumptions A and B.

Example 1: Let $\sigma \in \mathbb{R}$ be such that $0 < \sigma < 1$. Consider $r \in \mathbb{R}$ such that

$$\frac{\sigma}{1+\sigma} \le r < \frac{1}{2}. \tag{2.23}$$

For r fixed, let $s(x)$ be a smooth function on the interval $[r, 1-r]$ satisfying

$$\begin{aligned} &s(r) = 1, \quad s(1-r) = 0, \\ &s^{(t)}(r) = s^{(t)}(1-r) = 0, \text{ for } t = 1, 2, \ldots, l \end{aligned}$$

We now define a smooth function $\phi(x)$ on $[-1, 1]$ by

$$\phi(x) = \begin{cases} 1, & |x| \le r \\ 0, & |x| \ge 1-r \\ s(x), & r \le x \le 1-r \\ 1 - s(x+1), & -(1-r) \le x \le -r \end{cases}$$

Clearly, $\phi(x) \in C^l(-1,1)$ and support of $\phi(x)$ is $[-(1-r), (1-r)]$. We also note that

$$\phi(x) + \phi(x-1) = s(x) + 1 - s(x) = 1, \quad \text{for } r \le x \le 1-r. \tag{2.24}$$

Suppose $\Omega = (0,1)$ and we consider the nodes $x_i = ih$, $i = 0, 1, 2, \ldots, N$, where $Nh = 1$. For $i = 1, 2, \ldots, N-1$, we define patches

$$\begin{aligned} \omega_i &= (x_i - (1-r)h, x_i + (1-r)h), \\ \omega_i^* &= (x_i - rh, x_i + rh) \end{aligned} \tag{2.25}$$

For $i = 0, N$, we define

$$\begin{aligned} \omega_0 = (0, (1-r)h), &\qquad \omega_0^* = (0, rh) \\ \omega_N = (1-(1-r)h, 1), &\qquad \omega_N^* = (1-rh, 1) \end{aligned} \tag{2.26}$$

Clearly, $\cup_{i=0}^{N} \omega_i = \Omega$ and $\omega_i^* \subset \omega_i$ for $i = 0, 1, 2, \ldots, N$. Also using (2.23), we can easily show that

$$\frac{d_i^*}{d_i} = \frac{r}{1-r} > \sigma, \text{ for } i = 0, 1, \ldots, N,$$

where $d_i = \text{diam}(\omega_i)$ and $d_i^* = \text{diam}(\omega_i^*)$. Thus the patches $\{\omega_i\}$ satisfy Assumption A.

Now for each i, $0 \le i \le N$, define functions $\phi_i(x)$ on Ω by

$$\phi_i(x) = \begin{cases} \phi\left(\frac{x-x_i}{h}\right), & x \in [x_i - h, x_i + h] \\ 0, & x \in \Omega \setminus [x_i - h, x_i + h] \end{cases} \tag{2.27}$$

It is easy to check that $\phi_i(x) = 0$ for $x \in \Omega \setminus \omega_i$, and $\phi_i(x) = 1$ for $x \in \omega_i^*$ (which is Assumption B). Also from a standard scaling argument, it is immediate that

$$|\phi_i^{(t)}(x)| \le \frac{C_t}{|\text{diam}(\omega_i)|^t}, \quad 0 \le t \le l,$$

where C_t depends only on $\max_{y \in [-1,1]} |\phi^{(t)}(y)|$.

We now show that $\{\phi_j\}$ form a partition of unity. Consider the interval

$$[x_i, x_{i+1}] = [x_i, x_i + rh] \cup [x_i + rh, x_i + (1-r)h] \cup [x_i + (1-r)h, x_{i+1}].$$

For $x \in [x_i, x_i + rh]$, we have $0 \le (x - x_i)/h \le r$, and from the definition of ϕ_i and ϕ,

$$\sum_{j=0}^{N} \phi_j(x) = \phi_i(x) = \phi\left(\frac{x - x_i}{h}\right) = 1.$$

Similarly, for $x \in [x_i + (1-r)h, x_{i+1}]$,

$$\sum_{j=0}^{N} \phi_j(x) = \phi_{i+1}(x) = 1.$$

For $x \in [x_i + rh, x_i + (1-r)h]$, we have $r \le (x - x_i)/h \le 1 - r$, and from (2.24),

$$\begin{aligned} \sum_{j=0}^{N} \phi_j(x) &= \phi_i(x) + \phi_{i+1}(x) \\ &= \phi\left(\frac{x - x_i}{h}\right) + \phi\left(\frac{x - x_i}{h} - 1\right) = 1 \end{aligned}$$

Thus

$$\sum_{j=0}^{N} \phi_j(x) = 1, \quad \text{for all } x \in \Omega.$$

We note that for a two dimensional domain $\Omega = (0,1) \times (0,1)$, it is possible to construct patches of the form $\omega_i \times \omega_j$ and the partition of unity function of the form $\phi_i(x)\phi_j(y)$ that satisfy Assumptions A and B. We do not describe this construction in detail here.

Example 2: Let Ω be a domain in $\mathbb{R}^2$. For $0 < h < 1$, we consider the points $\{\mathbf{x}_j^h \equiv (x_j^h, y_j^h)\}_{j=1}^N$ such that $\mathbf{x}_j^h \in \overline{\Omega}$. We will suppress h in $\mathbf{x}_j^h$, and instead will denote these points by $\mathbf{x}_j$. We assume that the points are distributed in a way such that the following hold:

(a) For each j, there is a disc O_j^* of radius $r_j^* = O(h)$ and centered at $\mathbf{x}_j$ such $O_j^* \cap O_i^* = \emptyset$ for $i \neq j$. Let ω_j^* be the disc centered at $\mathbf{x}_j$ of radius $r_j^*/2$. If $\omega_j^* \not\subset \Omega$, we consider ω_j^{**} instead of ω_j^*, where $\omega_j^{**} = \omega_j^* \cap \Omega$ satisfies (2.18) with $b_j = \omega_j^*$ and $\omega_j^* = \omega_j^{**}$. We redefine ω_j^* as ω_j^{**}.

(b) For each j, there is a convex open set (*patch*) ω_j with $\text{diam}(\omega_j) = d_j = O(h)$ such that $O_j^* \subset \omega_j$ and there exists $0 < \sigma < 1$ such that

$$r_j^* \geq \sigma d_j, \quad 1 \leq j \leq N.$$

Moreover, $\{\omega_j\}_{j=1}^N$ form an open cover of Ω satisfying (2.4). It is easy to check that Assumption A is satisfied.

We will now construct partition of unity functions $\{\phi_j\}$, subordinate to the covering $\{\omega_j\}$, satisfying Assumption B. For each j, we first consider a smooth non-negative function $0 \leq \psi_j(x,y) \leq 1$ on Ω such that

$$(i) \qquad \psi_j(x,y) = \begin{cases} 0, & (x,y) \in \omega_j^* \\ 1, & (x,y) \in \Omega \backslash O_j^* \end{cases} \tag{2.28}$$

$$(ii) \qquad \max_{(x,y)\in\Omega} |D^\alpha \psi_j| \leq C/h^{|\alpha|}, \quad \text{for } |\alpha| \leq 2. \tag{2.29}$$

The function ψ_j could be a radial function based on a scaled and suitably defined one-dimensional function in $[0,\infty)$. We then define the function

$$\psi(x,y) = \prod_{j=1}^N \psi_j(x,y).$$

Clearly, $0 \leq \psi(x,y) \leq 1$ and using (2.28), (2.29), we can show that

$$\psi(x,y) = \begin{cases} 0, & (x,y) \in \omega_j^*, \ 1 \leq j \leq N \\ 1, & (x,y) \in \Omega \backslash \cup_{j=1}^N O_j^* \\ \psi_j(x,y), & (x,y) \in O_j^*, \ 1 \leq j \leq N \end{cases} \tag{2.30}$$

$$\max_{(x,y)\in\Omega} |D^\alpha \psi| \leq C/h^{|\alpha|}, \quad \text{for } |\alpha| \leq 2 \tag{2.31}$$

We next consider smooth non-negative functions $f_j(x,y)$ with compact support in the patch ω_j, satisfying $\max_{(x,y)\in\omega_j} |D^\alpha f_j| \leq C/h^{|\alpha|}$ for $|\alpha| \leq 2$. The functions $f_j(x,y)$ could be constructed as radial functions with circular support. Also, the construction of functions $f_j(x,y)$ with polygonal support have been discussed in [16, 15]. We further assume that there exists $\gamma > 0$ such that

$$\sum_{j=1}^N f_j(x,y) \geq \gamma > 0, \quad \text{for } (x,y) \in \Omega, \tag{2.32}$$

and define

$$\tilde{\phi}_j(x,y) = f_j(x,y)\psi(x,y) + \big[1 - \psi_j(x,y)\big], \quad 1 \leq j \leq N.$$

We now state some relevant properties of $\tilde{\phi}_j(x,y)$.

(i) The supp $\tilde{\phi}_j(x, y) \subset \omega_j$.

This is clear from the fact that supp$f_j(x, y) \subset \omega_j$ and $\psi_j(x, y) = 1$ for $(x, y) \in \Omega \backslash O_j^*$ (see (2.28).

(ii) $\tilde{\phi}_j(x, y) = \delta_{ij}$ for $(x, y) \in \omega_i^*$).

For $(x, y) \in \omega_j^*$, we have from (2.28) and (2.30), that $\psi_j(x, y) = \psi(x, y) = 0$ and thus $\tilde{\phi}_j(x, y) = 1$. Similarly, we show that $\tilde{\phi}_j(x, y) = 0$ for $(x, y) \in \omega_i^*$, $i \neq j$.

(iii) $\max_{(x,y)\in\omega_j} |D^\alpha \tilde{\phi}_j| \leq C/h^{|\alpha|}$, for $|\alpha| \leq 2$.

This is obtained using (2.29), (2.31), and the fact that $\max_{(x,y)\in\omega_j} |D^\alpha f_j| \leq C/h^{|\alpha|}$ for $|\alpha| \leq 2$.

(iv) There is $\tilde{\gamma} > 0$ such that $\sum_{j=1}^N \tilde{\phi}_j(x, y) \geq \tilde{\gamma}$, for $(x, y) \in \Omega$.

To obtain this result, we first note from (ii) that for $1 \leq i \leq N$ and for $(x, y) \in \omega_i^*$, we have $\tilde{\phi}_i(x, y) = 1$ and $\tilde{\phi}_j(x, y) = 0$ for $j \neq i$. Thus $\sum_{j=1}^N \tilde{\phi}_j(x, y) = \tilde{\phi}_i(x, y) = 1$.

Also for $(x, y) \in \Omega \backslash \cup_{j=1}^N O_j^*$, we have from (2.28) and (2.30) that $\psi(x, y) = 1$, $\psi_j(x, y) = 1$ for all j, and thus from (2.32) we get $\sum_{j=1}^N \tilde{\phi}_j(x, y) \geq \gamma$.

Moreover, for $(x, y) \in O_i^* \backslash \omega_i^*$, $1 \leq i \leq N$, we have $\psi_j(x, y) = 1$ for $j \neq i$ and consequently, using (2.32) and assuming $\gamma < 1$, we get

$$\begin{aligned} \sum_{j=1}^N \tilde{\phi}_j(x, y) &\geq \gamma\psi_i(x, y) + 1 - \psi_i(x, y) \\ &\geq 1 - |(\gamma - 1)|\psi_i(x, y) > 1 + \gamma - 1 = \gamma. \end{aligned}$$

If $\gamma \geq 1$, it is easy to show that $\sum_{j=1}^N \tilde{\phi}_j(x, y) \geq 1$ for $(x, y) \in O_i^* \backslash \omega_i^*$, $1 \leq i \leq N$.

Thus the result is true with $\tilde{\gamma} = \min(1, \gamma)$.

Finally, we use the technique of Shepard ([21, 31]) to define

$$\phi_j(x, y) = \frac{\tilde{\phi}_j(x, y)}{\sum_{i=1}^N \tilde{\phi}_i(x, y)}.$$

Using the properties (i)–(iv) of $\tilde{\phi}_j(x, y)$, given above, it is easy to check that $\{\phi_j\}_{j=1}^N$ is a partition of unity subordinate to the cover $\{\omega_j\}_{j=1}^N$ satisfying (2.6)-(2.9) and Assumption B.

The points $\{\mathbf{x}_j\}_{j=1}^N$ considered in this example have to be distributed such that (a) and (b), mentioned above, are satisfied. For example, if the particles $\mathbf{x}_j$ are vertices of a quasi-uniform triangulation of Ω, it is possible to construct ω_j^*, O_j^*, and ω_j satisfying (a) and (b).

3 Interior Estimate

Interior estimates play a crucial role in the study of superconvergence in a Galerkin method. In a series of papers ([25, 29, 30]), Nitsche, Schatz, and Wahlbin developed a machinery to establish interior estimates in the context of finite element method. This theory, which is based on certain axioms on the finite dimensional approximating subspace, can also be used in the context of GFEM. In this section, we will show that the finite dimensional space used in GFEM, *i.e.*, $S^h \equiv S^{GFEM}$, satisfies the axioms given in [30].

We first state the interior estimate that we will use in the next section. Let $\Omega_0 \subset\subset \Omega_D \subset\subset \Omega$ be domains, where $D = \text{dist}(\Omega_0, \partial\Omega_D)$. We also assume that all the patches ω_i's in a neighborhood of Ω_D are quasi-uniform, *i.e.*, $0 < \beta \leq d_i/h$. We also assume that $D \geq c_0 h$ for c_0 large enough. Let $u_h \in S^h(\Omega_D) \equiv S^{GFEM}(\Omega_D)$ be such that

$$B(u - u_h, v) = 0, \quad \text{for all } v \in \mathring{S}^h(\Omega_D). \tag{3.1}$$

Here $S^h(\Omega_D)$ denotes the restrictions of functions in $S^h(\Omega)$ to Ω_D, and $\mathring{S}^h(\Omega_D)$ denotes the set of functions in $S^h(\Omega_D)$ with compact support in the interior of Ω_D. We now state Theorem 1.2 from [30], which will be used later in this paper.

Theorem 3.1 ***(Theorem 1.2 of [30])*** *There exists a constant C, depending only on the constants in the Axioms A1–A5 (given below) over Ω_D, such that if $e = u - u_h$ satisfies (3.1), then*

$$\begin{aligned} &|e|_{W^1_\infty(\Omega_0)} + D^{-1}\|e\|_{L_\infty(\Omega_0)} \\ &\qquad \leq C \min_{\chi \in S^h} \left(|u - \chi|_{W^1_\infty(\Omega_D)} + D^{-1}\|u - \chi\|_{L_\infty(\Omega_D)}\right) \\ &\qquad\qquad + CD^{-2}\|e\|_{L_2(\Omega_D))}.\square \end{aligned} \tag{3.2}$$

Theorem 1.2 of [30] is quite general. The theorem, stated above, can be obtained by using $s = 0$, $q = 2$, and the fact that $\Omega \subset \mathbb{R}^2$ in Theorem 1.2 of [30] .

The above theorem holds provided the subspace $S^h(\Omega)$ satisfies certain axioms. For $G \subset \Omega$, let $S^h(G)$ be the restriction of $S^h(\Omega)$ to G, and let

$$\mathring{S}^h(G) = \{\chi \ : \ \chi \in S^h(G), \ \text{supp } \chi \subset\subset G\}.$$

Also, for $A \subset\subset \Omega$, define

$$\gamma(A) = \{j \in \mathbb{N} \ : \ A \cap \omega_j \neq \emptyset\}$$

and

$$\widetilde{A} = \bigcup_{j \in \gamma(A)} \omega_j.$$

It is clear that $A \subset \widetilde{A}$.

We now show that there exists positive constants C_1, C_2, C_3, C_4, k_0, and $h_0 < 1$ such that the space $S^h \equiv S^{GFEM}$ satisfies the following axioms.

Axiom A1. Approximation. Let $G_2 \subset G \subset\subset \Omega$ with $\text{dist}(G, \partial\Omega) \geq k_0 h$. Then for each $v \in W_q^l(G_2)$, there exists $\chi \in S^h(G)$ such that for $G_1 \subset\subset G_2$ with $\text{dist}(G_1, \partial G_2) \geq k_0 h$,

$$\|v - \chi\|_{W_q^t(G_1)} \leq C_1 h^{l-t} \|v\|_{W_q^l(G_2)} \tag{3.3}$$

for $0 \leq t \leq l \leq k+1$, $1 \leq q \leq \infty$, $t = 0, 1, 2$.

Moreover, if $\text{supp}\ v \subset G_1$, then $\chi \in \mathring{S}^h(G_2)$.

Remark 3.1 We note that in [30], the $W_q^{t,h}(G_1)$ norm was used in (3.3) instead of $W_q^t(G_1)$. It was natural to use $W_q^{t,h}(G_1)$ norm in [30], since the space S^h considered in [30] was a subset of $W_\infty^1 \cap C^{2,h}$, i.e., the functions is S^h were piecewise C^2 and globally W_∞^1 functions. We note that functions $\chi \in S^h \equiv S^{GFEM}$ are C^2 functions, and thus $\|\chi\|_{W_q^{2,h}(G_1)} = \|\chi\|_{W_q^2(G_1)}$. For a definition of $W_q^{t,h}(G_1)$ norm, we refer to page 925 of [30].

Proposition 3.1 *The subspace $S^h \equiv S^{GFEM}$ satisfies Axiom A1.*

Proof: Let $\bar{v}$ be an extension of v to Ω, such that $\bar{v} = v$ in G_2 and

$$\|\bar{v}\|_{W_q^l(\Omega)} \leq C \|v\|_{W_q^l(G_2)}. \tag{3.4}$$

For existence of such an extension, we refer to [32].

Let $\xi_j^{\bar{v}}$ be the averaged Taylor polynomial of $\bar{v}$ of degree $l-1$, averaged over ω_j^*. Then $\xi_j^{\bar{v}}\big|_{\omega_j} \in V_j$, and from Lemma 4.3.8 of [12], we know that

$$|\bar{v} - \xi_j^{\bar{v}}|_{W_q^t(\omega_j)} \leq C {d_j}^{l-t} |\bar{v}|_{W_q^l(\omega_j)}. \tag{3.5}$$

Define

$$\chi = \sum_{j=1}^N \phi_j \xi_j^{\bar{v}}.$$

Clearly, $\chi \in S^h(\Omega)$ and

$$|\bar{v} - \chi|^q_{W_q^t(\Omega)} \leq C \sum_{j=1}^N |\phi_j(\bar{v} - \xi_j^{\bar{v}})|^q_{W_q^t(\omega_j)},$$

where C depends on κ and q. Therefore, using (2.9), (3.5), and (3.4), we get

$$\begin{aligned} |\bar{v}-\chi|^q_{W^t_q(\Omega)} &\leq C\sum_{j=1}^{N}\sum_{i=0}^{t}|\phi_j|^q_{W^i_\infty(\omega_j)}\,|\bar{v}-\xi^{\bar{v}}_j|^q_{W^{t-i}_q(\omega_j)} \\ &\leq C\sum_{j=1}^{N}\sum_{i=0}^{t}(d_j)^{-qi}(d_j)^{(l-t+i)q}|\bar{v}|_{W^l_q(\omega_j)} \\ &\leq C\sum_{j=1}^{N}d_j^{\,(l-t)q}|\bar{v}|^q_{W^l_q(\omega_j)} \\ &\leq Ch^{(l-t)q}|\bar{v}|^q_{W^l_q(\Omega)} \leq Ch^{(l-t)q}\|v\|^q_{W^l_q(G_2)}. \end{aligned}$$

Hence,

$$|v-\chi|^q_{W^t_q(G_1)} \leq |\bar{v}-\chi|^q_{W^t_q(\Omega)} \leq Ch^{(l-t)q}\|v\|^q_{W^l_q(G_2)},$$

from which we get

$$\|v-\chi\|^q_{W^t_q(G_1)} \leq Ch^{(l-t)q}\|v\|^q_{W^l_q(G_2)}.$$

Now letting $\chi \equiv \chi|_G \in S^h(G)$ we get the (3.3).

We now suppose that supp $v \subset G_1$. We consider k_0 large enough that $\overline{\widetilde{G}}_1 \subset\subset G_2$. Since supp $v \subset G_1$, we have $\xi^{\bar{v}}_j = 0$ for $j \notin \gamma(G_1)$, where $\bar{v}$ is defined as the zero extension of v. Thus supp $\chi \subset \overline{\widetilde{G}}_1 \subset\subset G_2$, and hence, $\chi \in \mathring{S}^h(G_2)$, which proves the desired result. □

Axiom A2. Inverse inequality. Let $G_1 \subset\subset G_2$ with $\mathrm{dist}(G_1, \partial G_2) \geq k_0 h$. Then for $\chi \in S^h(G_2)$,

$$\|\chi\|_{H^1(G_1)} \leq C_2 h^{-1}\|\chi\|_{L_2(G_2)}. \tag{3.6}$$

Moreover,

$$\|\chi\|_{W^s_q(G_1)} \leq C_2 h^{t-s-2(1/q_1-1/q)}\|\chi\|_{W^t_{q_1}(G_2)}, \tag{3.7}$$

for $0 \leq t \leq s \leq 2$, $1 \leq q_1 \leq q \leq \infty$.

Remark 3.2 We note that (3.6) is a special case of a more general inverse inequality assumption given in [30]. A careful reading of the proof of Theorem 1.2 in [30] shows that we need only (3.6) to get Theorem 3.1 in this paper.

Proposition 3.2 *The subspace $S^h \equiv S^{GFEM}$ satisfies Axiom A2.*

Proof: Suppose $\chi \in S^h(G_2)$. Then

$$\chi\big|_{G_1} = \sum_{i\in\gamma(G_1)} \phi_i\xi_i, \quad \text{where } \xi_i \in V_i.$$

Recalling that $\phi_i = 0$ in $\Omega\backslash\omega_i$, we have

$$\begin{aligned}|\chi|^2_{H^1(G_1)} &\leq C \sum_{i\in\gamma(G_1)} |\phi_i\xi_i|^2_{H^1(G_1)} \\ &= C \sum_{i\in\gamma(G_1)} |\phi_i\xi_i|^2_{H^1(\omega_i)} \\ &\leq C \sum_{i\in\gamma(G_1)} \left(|\phi_i|^2_{W^1_\infty(\omega_i)}\|\xi_i\|^2_{L_2(\omega_i)} + \|\phi_i\|^2_{L_\infty(\omega_i)}|\xi_i|^2_{H^1(\omega_i)}\right),\end{aligned}$$

where C depends only on κ. Therefore from (2.8), (2.9), and (2.18) with $p = 2$, we have,

$$|\chi|^2_{H^1(G_1)} \leq Ch^{-2} \sum_{i\in\gamma(G_1)} \|\xi_i\|^2_{L_2(\omega_i)}. \tag{3.8}$$

We now consider the domain A such that $\widetilde{G}_1 \subset\subset A \subset\subset G_2$ for k_0 sufficiently large. Then from (2.17), we get

$$\begin{aligned}\|\chi\|^2_{L_2(A)} \geq \|\chi\|^2_{L_2(\widetilde{G}_1)} &\geq \sum_{i\in\gamma(G_1)} \|\chi\|^2_{L_2(\omega_i^*)} \\ &= \sum_{i\in\gamma(G_1)} \|\xi_i\|^2_{L_2(\omega_i^*)} \geq C \sum_{i\in\gamma(G_1)} \|\xi_i\|^2_{L_2(\omega_i)},\end{aligned}$$

and thus from (3.8), we have

$$\|\chi\|_{H^1(G_1)} \leq Ch^{-1}\|\chi\|_{L_2(A)} \leq Ch^{-1}\|\chi\|_{L_2(G_2)},$$

which is (3.6).

We now prove (3.7) for $s = 2$ and $t = 0$. The other cases can be proved similarly. Using the argument leading to (3.8), but employing (2.18) with $p = q$, and the fact that

$$\|\xi_i\|_{L_q(\omega_i)} \leq C(h^2)^{1/q-1/q_1}\|\xi_i\|_{L_{q_1}(\omega_i)}$$

we get,

$$\begin{aligned}|\chi|^q_{W^2_q(G_1)} &\leq Ch^{-2q} \sum_{i\in\gamma(G_1)} \|\xi_i\|^q_{L_q(\omega_i)} \\ &\leq Ch^{-2q}(h^2)^{1-q/q_1} \sum_{i\in\gamma(G_1)} \|\xi_i\|^q_{L_{q_1}(\omega_i)}.\end{aligned}$$

Therefore,

$$|\chi|_{W^2_q(G_1)} \leq Ch^{-2}(h^2)^{1/q-1/q_1} \left(\sum_{i\in\gamma(G_1)} \|\xi_i\|^q_{L_{q_1}(\omega_i)}\right)^{1/q},$$

and since $q_1 \le q$, we have

$$|\chi|_{W_q^2(G_1)} \le Ch^{-2} h^{2(1/q-1/q_1)} \left(\sum_{i\in\gamma(G_1)} \|\xi_i\|_{L_{q_1}(\omega_i)}^{q_1} \right)^{1/q_1}. \tag{3.9}$$

Again as before, but using (2.17) with $p = q_1$, we can show that

$$\|\chi\|_{L_{q_1}(G_2)}^{q_1} \ge C \sum_{i\in\gamma(G_1)} \|\xi_i\|_{L_{q_1}(\omega_i)}^{q_1},$$

and thus from (3.9) we get

$$|\chi|_{W_q^2(G_1)} \le Ch^{-2-2(1/q_1-1/q)} \|\chi\|_{L_{q_1}(G_2)} \tag{3.10}$$

Similarly we can show that

$$|\chi|_{W_q^j(G_1)} \le Ch^{-j-2(1/q_1-1/q)} \|\chi\|_{L_{q_1}(G_2)}, \quad \text{for } j = 0, 1,$$

and therefore using (3.10) we get

$$\|\chi\|_{W_q^2(G_1)} \le Ch^{-2-2(1/q_1-1/q)} \|\chi\|_{L_{q_1}(G_2)},$$

which is the desired result. □

Axiom A3. Superapproximation. Let $G_1 \subset\subset G_2 \subset\subset G_3$ with $\text{dist}(G_1, \partial G_2) \ge k_0 h$, $\text{dist}(G_2, \partial G_3) \ge k_0 h$ and let $\rho \in \mathring{C}^\infty(G_1)$. Then for each $\chi \in S^h(G_3)$, there exists an $\eta \in \mathring{S}^h(G_3)$ such that for some $\ell > 0$,

$$\|\rho\chi - \eta\|_{H^s(G_3)} \le C_3 h \|\rho\|_{W_\infty^\ell(G_1)} \|\chi\|_{H^s(G_2)}, \quad s = 0, 1, \tag{3.11}$$

and

$$\|\eta\|_{L_q(G_3)} \le C\|\chi\|_{L_q(G_3)}, \quad 1 \le q \le \infty. \tag{3.12}$$

Furthermore, let $G_{-2} \subset\subset G_{-1} \subset\subset G_0 \subset\subset G_1$ with $\text{dist}(G_{-2}, \partial G_{-1}) \ge k_0 h$, $\text{dist}(G_{-1}, \partial G_0) \ge k_0 h$ and $\text{dist}(G_0, \partial G_1) \ge k_0 h$. Then, if $\rho = 1$ on G_0, we have $\eta = \chi$ on G_{-1} and

$$\|\rho\chi - \eta\|_{H^s(G_3)} \le C_3 h \|\chi\|_{H^s(G_3\setminus G_{-2})}. \tag{3.13}$$

We first prove the following lemma.

Lemma 3.1 *Let ρ be a smooth function on ω_i and $\xi_i \in V_i$. Then there exists $\bar{\xi}_i \in V_i$ such that*

$$\|\rho\xi_i - \bar{\xi}_i\|_{H^s(\omega_i)} \le Ch\|\rho\|_{W_\infty^2(\omega_i)} \|\xi_i\|_{H^s(\omega_i)}, \quad s = 0, 1. \tag{3.14}$$

Proof: We prove (3.14) for $s = 1$. The proof for $s = 0$ is similar.

Recalling that V_i contains constants, we first decompose ξ_i as

$$\xi_i = \xi_{i1} + \xi_{i2},$$

where $\xi_{i1}, \xi_{i2} \in V_i$ and ξ_{i1} is a constant that is orthogonal to ξ_{i2} in $H^1(\omega_i)$ inner product, *i.e.*, $< \xi_{i1}, \xi_{i2} >_{H^1(\omega_i)} = 0$. Clearly, $\int_{\omega_i} \xi_{i2}\, dx = 0$ and from (2.14), we have

$$\|\xi_{i2}\|_{L_2(\omega_i)} \leq Ch|\xi_{i2}|_{H^1(\omega_i)}. \tag{3.15}$$

We will now construct $\bar{\xi}_i = \bar{\xi}_{i1} + \bar{\xi}_{i2} \in V_i$ such that (3.14) holds with $s = 1$. Let $L \in \mathcal{P}^1(\omega_i) \subset V_i$ such that

$$\|\rho - L\|_{W^1_\infty(\omega_i)} \leq Ch|\rho|_{W^2_\infty(\omega_i)} \tag{3.16}$$

(L could be the linear Taylor polynomial of ρ centered at the center of ω_i^*). We choose $\bar{\xi}_{i1} = \xi_{i1} L$. Clearly, $\bar{\xi}_{i1} \in V_i$, and using (3.16), we have

$$\begin{aligned} \|\rho\, \xi_{i1} - \bar{\xi}_{i1}\|_{H^1(\omega_i)} &= \|\xi_{i1}(\rho - L)\|_{H^1(\omega_i)} \\ &\leq C\|\rho - L\|_{W^1_\infty(\omega_i)} \|\xi_{i1}\|_{L_2(\omega_i)} \\ &\leq Ch|\rho|_{W^2_\infty(\Omega_i)} \|\xi_{i1}\|_{H^1(\omega_i)}. \end{aligned} \tag{3.17}$$

We next write $\rho(x) = \bar{\rho} + \rho^*(x)$, where $\bar{\rho}$ is a constant (we may take $\bar{\rho} = \rho(x_i)$, where x_i is the center of ω_i^*), and

$$\|\rho^*\|_{L_\infty(\omega_i)} \leq Ch\|\nabla\rho\|_{L_\infty(\omega_i)}. \tag{3.18}$$

We now choose $\bar{\xi}_{i2} = \bar{\rho}\xi_{i2}$. Clearly, $\bar{\xi}_{i2} \in V_i$ and using (3.15) and (3.18), we have

$$\begin{aligned} \|\rho\xi_{i2} - \bar{\xi}_{i2}\|_{H^1(\omega_i)} &= \|\rho^*\xi_{i2}\|_{H^1(\omega_i)} \\ &\leq C\|\rho^*\|_{L_\infty(\omega_i)} \|\xi_{i2}\|_{H^1(\omega_i)} + C\|\nabla\rho\|_{L_\infty(\omega_i)} \|\xi_{i2}\|_{L_2(\omega_i)} \\ &\leq Ch\|\nabla\rho\|_{L_\infty(\omega_i)} \|\xi_{i2}\|_{H^1(\omega_i)}. \end{aligned} \tag{3.19}$$

Finally, using (3.17), (3.19), and the fact that $< \xi_{i1}, \xi_{i2} >_{H^1(\omega_i)} = 0$, we have

$$\begin{aligned} \|\rho\xi_i - \bar{\xi}_i\|_{H^1(\omega_i)} &\leq \|\rho\xi_{i1} - \bar{\xi}_{i1}\|_{H^1(\omega_i)} + \|\rho\xi_{i2} - \bar{\xi}_{i2}\|_{H^1(\omega_i)} \\ &\leq Ch|\rho|_{W^2_\infty(\omega_i)} \|\xi_{i1}\|_{H^1(\omega_i)} + Ch\|\nabla\rho\|_{L_\infty(\omega_i)} \|\xi_{i2}\|_{H^1(\omega_i)} \\ &\leq Ch\|\rho\|_{W^2_\infty(\omega_i)} \|\xi_i\|_{H^1(\omega_i)}, \end{aligned}$$

which is the desired result. □

Proposition 3.3 *The subspace $S^h = S^{GFEM}$ satisfies Axiom A3.*

Proof: Suppose $\chi \in S^h(G_3)$. Since $\rho \in \overset{\circ}{C}{}^\infty(G_1)$, the function $\rho\chi$ has the form

$$\rho\chi = \sum_{i \in \gamma(G_1)} \rho\phi_i\xi_i, \quad \text{where } \xi_i \in V_i.$$

Let $P_i(\rho\xi_i) \in V_i$ be the H^1-projection of $\rho\xi_i$ onto V_i for $i \in \gamma(G_1)$. Then by Lemma 3.1, we have

$$\|\rho\xi_i - P_i(\rho\xi_i)\|_{H^1(\omega_i)} \leq \|\rho\xi_i - \bar{\xi}_i\|_{H^1(\omega_i)} \leq Ch\|\rho\|_{W^2_\infty(\omega_i)}\|\xi_i\|_{H^1(\omega_i)}, \quad i \in \gamma(G_1). \tag{3.20}$$

Also since $P_i(\rho\xi_i)$ is the H^1 projection onto V_i, and V_i contains constants, we have $\int_{\omega_i}[\rho\xi_i - P_i(\rho\xi_i)]\,dx = 0$, we have from (2.14),

$$\|\rho\xi_i - P_i(\rho\xi_i)\|_{L_2(\omega_i)} \leq Ch|\rho\xi_i - P_i(\rho\xi_i)|_{H^1(\omega_i)}. \tag{3.21}$$

We now define

$$\eta = \sum_{i\in\gamma(G_1)} \phi_i P_i(\rho\xi_i).$$

Clearly, $\eta \in S^h(G_3)$ and $\eta = 0$ in $\Omega\backslash\widetilde{G}_1$. We choose k_0 large enough such that $\overline{\widetilde{G}_1} \subset\subset G_2$. Thus $\eta \in \mathring{S}^h(G_3)$. Now

$$\begin{aligned}
\|\rho\chi - \eta\|^2_{H^1(G_3)} &= \|\rho\chi - \eta\|^2_{H^1(\widetilde{G}_1)} \\
&\leq C\sum_{i\in\gamma(G_1)} \|\phi_i(\rho\xi_i - P_i(\rho\xi_i))\|^2_{H^1(\widetilde{G}_1)} \\
&= C\sum_{i\in\gamma(G_1)} \|\phi_i(\rho\xi_i - P_i(\rho\xi_i))\|^2_{H^1(\omega_i)} \\
&\leq C\sum_{i\in\gamma(G_1)} \Big(\|\phi_i\|^2_{L_\infty(\omega_i)}|\rho\xi_i - P_i(\rho\xi_i)|^2_{H^1(\omega_i)} \\
&\qquad + \|\nabla\phi_i\|^2_{L_\infty(\omega_i)}\|\rho\xi_i - P_i(\rho\xi_i)\|^2_{L_2(\omega_i)}\Big)
\end{aligned} \tag{3.22}$$

Thus using (2.9), (3.21), and (3.20) in the above inequality and then using (2.17), we get

$$\begin{aligned}
\|\rho\chi - \eta\|^2_{H^1(G_3)} &\leq C\sum_{i\in\gamma(G_1)} |\rho\xi_i - P_i(\rho\xi_i)|^2_{H^1(\omega_i)} \\
&\leq Ch^2\|\rho\|^2_{W^2_\infty(G_1)} \sum_{i\in\gamma(G_1)} \|\xi_i\|^2_{H^1(\omega_i)} \\
&\leq Ch^2\|\rho\|^2_{W^2_\infty(G_1)} \sum_{i\in\gamma(G_1)} \|\xi_i\|^2_{H^1(\omega_i^*)}.
\end{aligned} \tag{3.23}$$

Since $\widetilde{G}_1 \subset G_2$, we get

$$\|\chi\|^2_{H^1(G_2)} \geq \sum_{i\in\gamma(G_1)} \|\chi\|^2_{H^1(\omega_i^*)} = \sum_{i\in\gamma(G_1)} \|\xi_i\|^2_{H^1(\omega_i^*)},$$

and combining it with (3.23) we have

$$\|\rho\chi - \eta\|_{H^1(G_3)} \leq Ch\|\rho\|_{W^2_\infty(G_1)}\|\chi\|_{H^1(G_2)}$$

which is (3.11) for $s = 1$. The case for $s = 0$ can be proved similarly.

We now prove (3.12). We first note from (3.20) and (2.18) that

$$\begin{aligned}\|P_i(\rho\xi_i)\|_{L_2(\omega_i)} &\le \|\rho\xi_i\|_{L_2(\omega_i)} + Ch\|\rho\|_{W^2_\infty(\omega_i)}\|\xi_i\|_{H^1(\omega_i)} \\ &\le \|\rho\|_{L_\infty(\omega_i)}\|\xi_i\|_{L_2(\omega_i)} + C\|\xi_i\|_{L_2(\omega_i)} \\ &\le C\|\xi_i\|_{L_2(\omega_i)}.\end{aligned}$$

Thus using (2.17) and the above, we get

$$\begin{aligned}\|\eta\|^2_{L_2(G_3)} &\le C\sum_{i\in\gamma(G_1)} \|\phi_i\, P_i(\rho\xi_i)\|^2_{L_2(\omega_i)} \\ &\le C\sum_{i\in\gamma(G_1)} \|P_i(\rho\xi_i)\|^2_{L_2(\omega_i)} \\ &\le C\sum_{i\in\gamma(G_1)} \|\xi_i\|^2_{L_2(\omega_i)} \\ &\le C\sum_{i\in\gamma(G_1)} \|\xi_i\|^2_{L_2(\omega_i^*)} \end{aligned} \tag{3.24}$$

Now,

$$\|\chi\|^2_{L_2(G_3)} \ge \sum_{i\in\gamma(G_1)} \|\chi\|^2_{L_2(\omega_i^*)} = \sum_{i\in\gamma(G_1)} \|\xi_i\|^2_{L_2(\omega_i^*)},$$

and therefore from (3.24), we have

$$\|\eta\|_{L_2(G_3)} \le C\|\chi\|_{L_2(G_3)}.$$

Finally, using the fact that $S^h(G_3)$ is finite dimensional, we get

$$\|\eta\|_{L_q(G_3)} \le C\|\chi\|_{L_q(G_3)}, \quad \text{for } 1 \le q \le \infty.$$

We now prove (3.13). We first assume that $\text{dist}(G_{-1}, \partial G_0) \ge k_0 h$ for a suitable k_0 such that $\widetilde{G}_{-1} \subset G_0$. Since $\rho = 1$ on G_0, we have $\rho = 1$ on ω_i for $i \in \gamma(G_{-1})$. Also from the definition of H^1 projection, we have $P_i(\rho\xi_i) = P_i(\xi_i) = \xi_i$ for $i \in \gamma(G_{-1})$. Therefore, for $x \in G_{-1}$,

$$\eta(x) = \sum_{i\in\gamma(G_1)} \phi_i(x)P_i(\rho\xi_i)(x) = \sum_{i\in\gamma(G_1)} \phi_i(x)\xi_i(x) = \chi(x).$$

Thus $\eta = \chi$ on G_{-1}.

Now using the argument leading to (3.22) and using (3.21) and (2.9) we have,

$$\|\rho\chi - \eta\|^2_{H^1(G_3)} \le C\sum_{i\in\gamma(G_1)} |\rho\xi_i - P_i(\rho\xi_i)|^2_{H^1(\omega_i)}. \tag{3.25}$$

We note that $\gamma(G_{-1}) \subset \gamma(G_1)$. Also recall that $\rho = 1$ and $P_i(\rho\xi_i) = \xi_i$ on ω_i for $i \in \gamma(G_{-1})$. Thus from (3.25), (3.20), and (2.17), we get

$$\begin{aligned} \|\rho\chi - \eta\|^2_{H^1(G_3)} &\leq C \sum_{i\in\gamma(G_1)\setminus\gamma(G_{-1})} |\rho\xi_i - P_i(\rho\xi_i)|^2_{H^(\omega_i)} \\ &\leq Ch^2\|\rho\|^2_{W^2_\infty(G_1)} \sum_{i\in\gamma(G_1)\setminus\gamma(G_{-1})} \|\xi_i\|^2_{H^1(\omega_i)} \\ &\leq Ch^2\|\rho\|^2_{W^2_\infty(G_1)} \sum_{i\in\gamma(G_1)\setminus\gamma(G_{-1})} \|\xi_i\|^2_{H^1(\omega_i^*)}. \end{aligned} \tag{3.26}$$

Now,

$$\|\chi\|^2_{H^1(G_3\setminus G_{-2})} \geq \sum_{i\in\gamma(G_1)\setminus\gamma(G_{-1})} \|\chi\|^2_{H^1(\omega_i^*)} = \sum_{i\in\gamma(G_1)\setminus\gamma(G_{-1})} \|\xi_i\|^2_{H^1(\omega_i^*)},$$

and therefore from (3.26) we get

$$\|\rho\chi - \eta\|_{H^1(G_3)} \leq Ch\|\rho\|_{W^2_\infty(G_1)}\|\chi\|_{H^1(G_3\setminus G_{-2})},$$

which is the desired result. □

We remark that the Axiom A3 as stated in this paper is slightly different than the Axiom A3 given in [30]. We further remark that Axiom A3 in [30] has been used only to prove Lemma 2.3 (Page 911) in that paper. We now prove Lemma 2.3 of [30] using the Axiom A3 as stated in this paper. The proof is similar to the proof of Proposition 2.2 in [29].

Lemma 3.2 *Let $D_1 \subset\subset D_2 \subset\subset D_3$. There exists a constant C such that given $\chi \in S^h(D_3)$, there exists $\eta \in \mathring{S}^h(D_3)$ with $\eta = \chi$ on D_2 such that*

$$\|\chi - \eta\|_{H^1(D_3\setminus D_2)} \leq C\|\chi\|_{H^1(D_3\setminus D_1)}$$

and

$$\|\eta\|_{L_q(D_3)} \leq C\|\chi\|_{L_q(D_3)}, \quad \textit{for } 1 \leq q \leq \infty.$$

Proof: Let $D_1 \subset\subset D_2 \subset\subset D_{21} \subset\subset D_{22} \subset\subset D_{23} \subset\subset D_3$. Consider $\rho \in \mathring{C}^\infty(D_{22})$ such that $\rho = 1$ on D_{21}. Then from (3.13) and (3.12), with $D_3 = G_3$, $D_{23} = G_2$, $D_{22} = G_1$, $D_{21} = G_0$, $D_2 = G_{-1}$, and $D_1 = G_{-2}$, there exists $\eta \in \mathring{S}^h(D_3)$ with $\eta = \chi$ on D_2 such that

$$\|\rho\chi - \eta\|_{H^1(D_3)} \leq Ch\|\chi\|_{H^1(D_3\setminus D_1)}, \tag{3.27}$$

and

$$\|\eta\|_{L_q(D_3)} \leq C\|\chi\|_{L_q(D_3)}.$$

Since $\rho = 1$ on D_{21}, we have

$$\|(1-\rho)\chi\|_{H^1(D_3)} \leq C\|\chi\|_{H^1(D_3\setminus D_{21})} \leq C\|\chi\|_{H^1(D_3\setminus D_1)},$$

and thus using the triangle inequality and (3.27), we have

$$\|\chi - \eta\|_{H^1(D_3)} \le \|(1-\rho)\chi\|_{H^1(D_3)} + \|\rho\chi - \eta\|_{H^1(D_3)} \le C\|\chi\|_{H^1(D_3\setminus D_1)}$$

Finally, since $\eta = \chi$ on D_2, we have

$$\|\chi - \eta\|_{H^1(D_3\setminus D_2)} = \|\chi - \eta\|_{H^1(D_3)} \le C\|\chi\|_{H^1(D_3\setminus D_1)}$$

which is the desired result. □

Axiom A4. Scaling. Let the sets G in Axiom A1, G_2 in Axiom A2, G_3 in Axiom A3 be the sphere $B_D \subset\subset \Omega$ of radius $D \ge C_4 h$ with center x_0. The linear transformation $y = (x - x_0)/D$ takes B_D into a sphere B and $S^h(B_D)$ into a new function space $S(B)$. Then $S(B)$ satisfies Axioms A1, A2 and A3 with h replaced by h/D. Furthermore, the constants occurring in Axioms A1, A2, and A3 remain unchanged, in particular, independent of D.

Using a standard scaling argument argument, one can show that Axiom A4 holds with respect to Axioms A1 and A2. To show that Axiom A4 holds with respect to Axiom A3, one has to go through the proof of Axiom A3 with $S^h(B_D)$ replaced by $S(B)$. We skip this proof in this paper.

Axiom A5. There exists a constant C_5 such that the following holds:

(i) For any $x_0 \in \Omega$ such that the ball B_0 of radius h centered at x_0 is contained in Ω, there exists a function $\widetilde{\delta}_0 \in C^1$ with support in B_0 satisfying

$$\chi(x_0) = \int_{B_0} \chi\widetilde{\delta}_0, \quad \text{for all } \chi \in S^h,$$

and

$$\|\widetilde{\delta}_0\|_{L_q} \le C_5 h^{-N(1-1/q)}, \; \|\nabla\widetilde{\delta}_0\|_{L_q} \le C_5 h^{-N(1-1/q)-1}, \text{ for } 1 \le q \le \infty. \quad (3.28)$$

(ii) Similarly, for $j = 1, 2, \ldots, N$, there exists $\widetilde{\delta}_{1,j}$ such that

$$\frac{\partial\chi}{\partial x_j}(x_0) = \int_{B_0} \frac{\partial\chi}{\partial x_j}\,\widetilde{\delta}_{1,j}\;, \quad \text{for all } \chi \in S^h,$$

and (3.28) holds with $\widetilde{\delta}_0$ replaced by $\widetilde{\delta}_{1,j}$.

We will show the existence $\widetilde{\delta}_{1,j}$. The existence of $\widetilde{\delta}_0$ can be shown similarly. We let $h = 1$. Let Φ be a smooth non-negative weight function with compact support in B_0 and suppose

$$\int_{B_0} \Phi\, dx = 1.$$

Consider the inner product

$$< v, w > \equiv \int_{B_0} \Phi\, v\, w\, dx,$$

and let $\psi_1, \psi_2, \ldots, \psi_\ell$ be an orthonormal basis, with respect to this inner product, for the finite dimensional space $\{\frac{\partial \chi}{\partial x_j} : \chi \in S^h(B_0)\}$. Define

$$\widetilde{\delta}_{1,j} \equiv \sum_{j=1}^{\ell} \psi_j(x_0)\psi_j(x)\Phi(x).$$

Let $\chi \in S^h$ such that $\frac{\partial \chi}{\partial x_j} = \sum_{m=1}^{\ell} c_m \psi_m$. Then

$$\begin{aligned}\int_{B_0} \frac{\partial \chi}{\partial x_j} \widetilde{\delta}_{1,j}\, dx &= \sum_{j=1}^{\ell} \psi_j(x_0) \int_{B_0} [\sum_{m=1}^{\ell} c_m \psi_m] \psi_j \Phi\, dx \\ &= \sum_{j=1}^{\ell} \psi_j(x_0) c_j = \frac{\partial \chi}{\partial x_j}(x_0).\end{aligned}$$

We get (3.28) by scaling.

Remark 3.3 We mention that another Axiom A0, related to a trace inequality, was used in [30]. It was relevant there since the functions in S^h, considered in [30], were piecewise C^2 and globally W^1_∞. In our case, the functions in $S^h = S^{GFEM}$ are globally C^2 and thus, a trace inequality like Axiom A0 in [30] is not needed.

4 Superconvergence

In this section we will present the main result of this paper. *i.e.*, the natural superconvergence of the derivatives of the Generalized Finite Element solution in the interior of the domain Ω, away from the boundary of Ω. The analysis presented in this section will closely follow the analysis given in [10], [35] in the context of finite element method. We will need several other assumptions that will be stated in this section.

Without loss of generality, we assume that $x_0 \equiv (0,0) \in \Omega$ and

$$\Omega_0 = \big\{x = (x_1, x_2) \in \Omega : \|x\|_\infty \equiv \max(|x_1|, |x_2|) \le 2H\big\} \subset\subset \Omega, \tag{4.1}$$

where $H > 0$ will be determined later. We also define the set

$$\Omega_1 = \big\{x \in \Omega : \|x\|_\infty \le H\big\}. \tag{4.2}$$

It is clear that the solution $u_h = u_{GFEM}$ of (2.12) satisfies

$$B(u - u_h, \chi) = 0, \quad \text{for all } \chi \in \mathring{S}^h(\Omega_0), \tag{4.3}$$

where we recall that $\mathring{S}^h(\Omega_0)$ denotes the restrictions of the functions in $S^h(\Omega)$ with compact support in the interior of Ω_0.

We now describe several assumptions in ***SC1*** – ***SC4*** that will be used in the analysis presented in this section.

SC1: We assume that the patches ω_i, such that $\omega_i \cap \Omega_0 \neq \emptyset$, are uniform and translation invariant as defined below. Consider $\bar{h} > 0$, where $\bar{h}$ is of the same order as h (recall that $\text{diam}(\omega_i) \leq 2h$), i.e., there exists a positive constant C such that

$$\bar{h} = Ch.$$

Let $i \equiv (i_1, i_2)$, where $i_1, i_2 = 0, \pm 1, \pm 2, \cdots$, be the integer lattice. Suppose $x_i \equiv i\bar{h} = (i_1\bar{h}, i_2\bar{h})$ is the center of the circle $\omega_i^* \subset \omega_i$ for $i \in \gamma(\Omega_0)$, where

$$\gamma(\Omega_0) = \big\{ i \in \mathbb{Z} \times \mathbb{Z} : \omega_i \cap \Omega_0 \neq \emptyset \big\}$$

We note that we are enumerating ω_i differently from the way we enumerated them in (2.5). Clearly, $x_{i+j} = x_i + x_j$ for $i, j, i+j \in \gamma(\Omega_0)$. We further assume that for all $i \in \gamma(\Omega_0)$,

$$\begin{aligned} \omega_i &= \{x + x_i : x \in \omega_0\}, && (4.4)\\ \omega_i^* &= \{x + x_i : x \in \omega_0^*\}, && (4.5)\\ \phi_i(x) &= \phi_0(x - x_i),\ x \in \omega_i, && (4.6)\\ V_i &= \big\{\xi_i(x) : \xi_i(x) = \xi_0(x - x_i),\ \text{where } \xi_0 \in V_0\big\}. && (4.7) \end{aligned}$$

Clearly,

$$\phi_i(x - x_j) = \phi_{i+j}(x), \quad \text{for } i, j, i+j \in \gamma(\Omega_0) \tag{4.8}$$

and

$$V_{i+j} = \big\{\xi_i(x - x_j) : \xi_i \in V_i\big\}, \quad \text{for } i, j, i+j \in \gamma(\Omega_0) \tag{4.9}$$

SC2: We consider H in (4.1) and (4.2) of the form

$$H = \bar{h}^{\beta} \tag{4.10}$$

where $0 < \beta < 1$ will be determined later. We let

$$M_0 = \big\{x \in \Omega : \|x\|_\infty \leq \bar{h}/2\big\} \tag{4.11}$$

and define

$$M_j = \big\{x \in \Omega : x = x_j + y,\ \text{where } y \in M_0\big\} \tag{4.12}$$

Clearly,

$$M_j = \big\{x \in \Omega : \|x - x_j\|_\infty \leq \bar{h}/2\big\} \tag{4.13}$$

and we assume the $\bar{h}$ and H are such that

$$\overline{\Omega}_0 = \bigcup_{x_j \in \Omega_0} M_j. \tag{4.14}$$

SC3: For a given H, let $\bar{H} > 0$ such that

$$\widetilde{\Omega}_0 = \bigcup_{j \in \gamma(\Omega_0)} \omega_j \subset \widetilde{\widetilde{\Omega}}_0,$$

where

$$\tilde{\tilde{\Omega}}_0 = \{x \in \Omega : \|x\|_\infty \le \bar{H}\} \tag{4.15}$$

and $\bar{H} \le CH$ where $1 < C$ is a fixed constant. We assume that the solution u of (2.4) is smooth in $\tilde{\tilde{\Omega}}_0$, i.e.,

$$\|u\|_{W_\infty^{k+2}(\tilde{\tilde{\Omega}}_0)} \le C \tag{4.16}$$

where k is the degree of the polynomials in $V_j = \mathcal{P}^k(\omega_j)$.

SC4: We assume that the solutions u and u_h of (2.4) and (2.12) satisfy

$$Ch^k \le |u - u_h|_{W_\infty^1(\Omega_1)} \tag{4.17}$$

$$\|u - u_h\|_{L_\infty(\Omega_0)} \le Ch^{k+1-\epsilon}, \tag{4.18}$$

where $0 < \epsilon < 1$.

We now define a function $\rho^h(x)$, which will play an important role in the analysis presented in this section. Towards this end, we first define a linear operator $I_i^h : W_\infty^{k+1}(\omega_i) \to V_i = \mathcal{P}^k(\omega_i)$ for each $i \in \gamma(\Omega_0)$ satisfying

$$(i)\ I_i^h[p_k(\cdot)](x) = p_k(x), \quad \text{for } x \in \omega_i, \text{ and for all } p_k \in \mathcal{P}^k \tag{4.19}$$

$$\begin{aligned}(ii)\ &\text{For } v \in W_\infty^{k+1}(\omega_i),\\ &\|v - I_i^h[v(\cdot)]\|_{W_q^l(\omega_i)} \le Cd_i^{k+1-l}\|v\|_{W_q^{k+1}(\omega_i)},\\ &\qquad \text{for all } 0 \le l \le k+1 \text{ and } 1 \le q \le \infty\end{aligned} \tag{4.20}$$

$$\begin{aligned}(iii)\ &\text{For } v \in W_\infty^{k+1}(\omega_{j+l}) \text{ and for all } l, l+j \in \gamma(\Omega_0)\\ &I_{l+j}^h[v(\cdot)](x + x_j) = I_l^h[v(\cdot + x_j)](x), \text{ for all } x \in \omega_l.\end{aligned} \tag{4.21}$$

For $I_i^h[v(\cdot)](x)$, one could take the restriction of the Taylor polynomial of $v(x)$, centered at x_i, to ω_i. We then define the operator $I^h : W_\infty^{k+1}(\tilde{\tilde{\Omega}}_0) \to S^h(\Omega_0)$ by

$$I^h[Q(\cdot)](x) = \sum_{i \in \gamma(\Omega_0)} \phi_i(x) I_i^h[Q(\cdot)](x), \quad x \in \Omega_0, \tag{4.22}$$

where $Q \in W_\infty^{k+1}(\tilde{\tilde{\Omega}}_0)$. Clearly, for a polynomial p_k of degree k,

$$\begin{aligned} I^h[p_k(\cdot)](x) &= \sum_{i \in \gamma(\Omega_0)} \phi_i(x) I_i^h[p_k(\cdot)](x) \\ &= \sum_{i \in \gamma(\Omega_0)} \phi_i(x) p_k(x) = p_k(x), \quad x \in \Omega_0 \end{aligned} \tag{4.23}$$

The operator I^h also satisfies the standard interpolation estimate given in the following lemma.

Lemma 4.1 *Let* $v \in W_\infty^{k+1}(\tilde{\tilde{\Omega}}_0)$. *Then*

$$\|v - I^h[v(\cdot)]\,\|_{W_q^t(\Omega_0)} \le Ch^{k+1-t}\|v\|_{W_q^{k+1}(\tilde{\tilde{\Omega}}_0)}, \text{ for } 1 \le q \le \infty \text{ and } t = 0, 1, 2. \tag{4.24}$$

Proof: We first note that for $x \in \Omega_0$,

$$v - I^h v = v - \sum_{i \in \gamma(\Omega_0)} \phi_i\, I_i^h[v(\cdot)] = \sum_{i \in \gamma(\Omega_0)} \phi_i\big(v - I_i^h[v(\cdot)]\big).$$

Therefore, using (2.6), (2.9), and (4.20), we have

$$\begin{aligned}
|v - I^h v|^q_{W_q^t(\Omega_0)} &\le C \sum_{i \in \gamma(\Omega_0)} \big|\phi_i\big(v - I_i^h[v(\cdot)]\big)\big|^q_{W_q^t(\omega_i)} \\
&\le C \sum_{i \in \gamma(\Omega_0)} \sum_{j=0}^{t} |\phi_i|^q_{W_\infty^j(\omega_i)} \big|v - I_i^h[v(\cdot)]\,\big|^q_{W_q^{t-j}(\omega_i)} \\
&\le C \sum_{i \in \gamma(\Omega_0)} d_i^{(k+1-t)q}\|v\|^q_{W_q^{k+1}(\omega_i)} \\
&\le Ch^{(k+1-t)q}\|v\|^q_{W_q^{k+1}(\tilde{\tilde{\Omega}}_0)},
\end{aligned}$$

where we used the fact that $d_i \le 2h$ in the last step. Thus we get, for small h,

$$\|v - I^h[v(\cdot)]\,\|_{W_q^t(\Omega_0)} \le Ch^{k+1-t}\|v\|_{W_q^{k+1}(\tilde{\tilde{\Omega}}_0)},$$

which is the desired result. □

We finally define $\rho^h(x)$ as

$$\rho^h(x) \equiv Q(x) - I^h[Q(\cdot)](x), \quad x \in \Omega_0, \tag{4.25}$$

where $Q(x)$ is a polynomial of degree $k+1$.

Lemma 4.2 $\rho^h(x)$ *is periodic in* $\overline{\Omega}_0$, *i.e.,*

$$\rho^h(x) = \rho^h(x + x_j), \quad \text{for } x \in M_0 \text{ and } x + x_j \in \overline{\Omega}_0.$$

Proof: We first note that

$$Q(x) = Q(x + x_j) - p_k(x; x_j), \tag{4.26}$$

where $p_k(x; x_j)$ is a polynomial of degree k that depends on x_j. Now from the definition of $\rho^h(x)$ and using (4.26), (4.23), we have

$$\begin{aligned}
\rho^h(x) &= Q(x) - I^h[Q(\cdot)](x) \\
&= Q(x + x_j) - p_k(x; x_j) - I^h[Q(\cdot)](x) \\
&= Q(x + x_j) - I^h[p_k(\cdot; x_j) + Q(\cdot)](x) \\
&= Q(x + x_j) - I^h[Q(\cdot + x_j)](x).
\end{aligned} \tag{4.27}$$

We will now show that

$$I^h[Q(\cdot + x_j)](x) = I^h[Q(\cdot)](x + x_j), \quad \text{for } x \in M_0 \text{ and } x + x_j \in \overline{\Omega}_0.$$

Since $x \in M_0$, we have $x + x_j \in M_j$, and using (4.21), we get

$$\begin{aligned} I^h[Q(\cdot)](x+x_j) &= \sum_{i\in\gamma(M_j)} \phi_i(x+x_j) I_i^h[Q(\cdot)](x+x_j) \\ &= \sum_{i\in\gamma(M_j)} \phi_{i-j}(x) I_i^h[Q(\cdot)](x+x_j) \\ &= \sum_{l\in\gamma(M_0)} \phi_l(x) I_{l+j}^h[Q(\cdot)](x+x_j) \\ &= \sum_{l\in\gamma(M_0)} \phi_l(x) I_l^h[Q(\cdot+x_j)](x) \end{aligned}$$

Thus from (4.27) we get

$$\rho^h(x) = Q(x+x_j) - I^h[Q(\cdot)](x+x_j) = \rho^h(x+x_j)$$

which is the desired result. □

The above result can also be stated as $\rho^h(x) \in H^1_{\text{per}}(\Omega_0)$, where

$$H^1_{\text{per}}(\Omega_0) = \big\{v \in H^1(\Omega_0) \cap C^0(\Omega_0) : v(x) = v(x+x_j) \text{ for } x \in M_0, x+x_j \in \Omega_0\big\}. \tag{4.28}$$

Next, for a given $v \in H^1_{\text{per}}(\Omega_0)$, we will define its periodic projection $P_{\text{per}}v \in S^h(\Omega_0) \cap H^1_{\text{per}}(\Omega_0)$, and present two results that will also be used in our analysis. We first consider the subspace

$$\begin{aligned} S^h_{\text{per}}(M_0) = \big\{\chi \in S^h(M_0) \quad : \quad & \chi(-\bar{h}/2, x_2) = \chi(\bar{h}/2, x_2), \\ & \chi(x_1, -\bar{h}/2) = \chi(x_1, \bar{h}/2), \\ & \text{for } |x_1| \le \bar{h}/2, |x_2| \le \bar{h}/2\big\}. \end{aligned} \tag{4.29}$$

For $v \in H^1_{\text{per}}(\Omega_0)$, we define $P_{\text{per}}v \in S^h_{\text{per}}(M_0)$ as the projection

$$B_{M_0}(P_{\text{per}}v, \chi) = B_{M_0}(v, \chi), \quad \text{for all } \chi \in S^h_{\text{per}}(M_0), \tag{4.30}$$

$$\int_{M_0} (v - P_{\text{per}}v)\, dx = 0, \tag{4.31}$$

where $B_{M_0}(u,v) \equiv \int_{M_0} \nabla u \cdot \nabla v \, dx$ (see 2.3)). We then extend $P_{\text{per}}v$ periodically to Ω_0, again denoted by $P_{\text{per}}v$, as

$$P_{\text{per}}v(x) = P_{\text{per}}v(x - x_j), \quad \text{for all } x \in \Omega_0, x - x_j \in M_0. \tag{4.32}$$

Thus $P_{\text{per}}v \in S^h(\Omega_0) \cap H^1_{\text{per}}(\Omega_0)$.

Lemma 4.3 *Let $v \in H^1_{\text{per}}(\Omega_0)$. Then*

$$B(v - P_{\text{per}}v, \chi) = 0, \quad \textit{for all } \chi \in \mathring{S}^h(\Omega_0). \tag{4.33}$$

Proof: Since $\mathring{M}_j$'s (i.e., interior of M_j) are non-intersecting and $\overline{\Omega}_0 = \cup_{x_j \in \Omega_0} M_j$, we note that

$$B(v - P_{\text{per}}v, \chi) = \sum_{x_j \in \Omega_0} B_{M_j}(v - P_{\text{per}}v, \chi). \tag{4.34}$$

Now using the periodicity of $v - P_{\text{per}}v$, we get

$$\begin{aligned} B_{M_j}(v - P_{\text{per}}v, \chi) &= \int_{M_j} \nabla(v - P_{\text{per}}v) \cdot \nabla\chi \, dx \\ &= \int_{M_0} \nabla\big(v(y + x_j) - P_{\text{per}}v(y + x_j)\big) \cdot \nabla\chi(y + x_j) \, dy \\ &= \int_{M_0} \nabla\big(v(y) - P_{\text{per}}v(y)\big) \cdot \nabla\chi(y + x_j) \, dy \end{aligned} \tag{4.35}$$

Thus from (4.34), we get

$$B(v - P_{\text{per}}v, \chi) = \int_{M_0} \nabla\big(v - P_{\text{per}}v\big) \cdot \nabla\hat{\chi} \, dy = B_{M_0}(v - P_{\text{per}}v, \hat{\chi}) \tag{4.36}$$

where

$$\hat{\chi}(y) = \sum_{x_j \in \Omega_0} \chi(y + x_j).$$

Since $\chi \in \mathring{S}^h(\Omega_0)$, we can show that $\hat{\chi} \in S^h_{\text{per}}(M_0)$. Thus using (4.30) in (4.36), we get

$$B(v - P_{\text{per}}v, \chi) = 0$$

which is the desired result. □

Lemma 4.4 *Let $v \in H^1_{\text{per}}(\Omega_0)$. Then*

$$\|v - P_{\text{per}}v\|_{L_2(\Omega_0)} \leq Ch\|v\|_{H^1(\Omega_0)}. \tag{4.37}$$

Proof: We first note that since $v - P_{\text{per}}v$ is periodic, using (4.31) we have

$$\int_{M_j} (v - P_{\text{per}}v) \, dx = \int_{M_0} (v - P_{\text{per}}v) \, dy = 0,$$

and using Poincare inequality, we get

$$\|v - P_{\text{per}}v\|_{L_2(M_j)} \leq C\bar{h}|v - P_{\text{per}}v|_{H^1(M_j)} \leq h|v - P_{\text{per}}v|_{H^1(M_j)}.$$

Therefore,

$$
\begin{aligned}
\|v - P_{\rm per}v\|^2_{L_2(\Omega_0)} &= \sum_{x_j \in \Omega_0} \|v - P_{\rm per}v\|^2_{L_2(M_j)} \\
&\le Ch \sum_{x_j \in \Omega_0} |v - P_{\rm per}v|^2_{H^1(M_j)} \\
&\le Ch \sum_{x_j \in \Omega_0} \left\{ |v|^2_{H^1(M_j)} + |P_{\rm per}v|^2_{H^1(M_j)} \right\} \\
&= Ch\left\{ |v|^2_{H^1(\Omega_0)} + |P_{\rm per}v|^2_{H^1(\Omega_0)} \right\}. \qquad (4.38)
\end{aligned}
$$

Again using the periodicity of v and $P_{\rm per}v$, and the fact that $P_{\rm per}v\big|_{M_0}$ is the projection of $v\big|_{M_0}$ onto $S^h_{\rm per}(M_0)$, we have

$$
|P_{\rm per}v|_{H^1(\Omega_0)} \le C|v|_{H^1(\Omega_0)},
$$

and thus from (4.38), we get the desired result. □

We now present our main theorem.

Theorem 4.1 *Suppose the assumptions* ***A1***–***A3*** *hold and the assumption* ***SC1***–***SC4*** *are satisfied with* $0 < \beta < 1 - \epsilon$*, where* β *and* ϵ *are as in (4.10) and (4.18) respectively. Also assume that* $\sum_{|s|=k+1} |D^s u(x_0)|^2 > 0$*, where* $s = (s_1, s_2)$ *is a multi-index and* $x_0 = (0, 0)$*. Then, for* h *small enough, there exists* $\alpha > 0$ *such that, for* $i = 1, 2$*,*

$$
\frac{\partial}{\partial x_i}(u - u_h)(x) = \frac{\partial}{\partial x_i}(\rho^h - P_{\rm per}\rho^h)(x) + R_i(x), \quad \textit{for } x \in \Omega_1, \qquad (4.39)
$$

where

$$
\|R_i\|_{L_\infty(\Omega_1)} \le Ch^{k+\alpha},
$$

and $Q(x)$ *in the definition of* $\rho^h(x)$ *(see (4.25)) is the* $(k+1)^{th}$ *degree Taylor polynomial of* u *centered at* x_0*.*

Remark 4.1 We note that we have assumed u to be smooth; in particular, u satisfies (4.16) in assumption ***SC4***.

Remark 4.2 If $x^* \in \Omega_1$ is a zero of $\frac{\partial}{\partial x_i}(\rho^h - P_{\rm per}\rho^h)$, then from the above result we get

$$
\left| \frac{\partial}{\partial x_i}(u - u_h)(x^*) \right| \le Ch^{k+\alpha},
$$

and thus x^* is a natural superconvergence point of $\frac{\partial}{\partial x_i}(u - u_h)$.

Proof of Theorem 4.1: The proof will be given in four steps.

1. Since Q is the $(k+1)^{th}$ degree Taylor polynomial of u centered at $x_0 = (0,0)$, we have

$$\begin{aligned}\|u-Q\|_{W^s_\infty(\Omega_0)} &\le \|u-Q\|_{W^s_\infty(\tilde{\tilde{\Omega}}_0)} \\ &\le C\bar{H}^{k+2-s} \le CH^{k+2-s}, \quad \text{for } 0 \le s \le k+2, \end{aligned} \tag{4.40}$$

where $\tilde{\tilde{\Omega}}_0$ is defined in (4.15). We now define the Neumann projection $P_N^h Q$ of Q onto $S^h(\Omega_0)$ by

$$B_{\Omega_0}(Q - P_N^h Q, \chi) = 0, \quad \text{for all } \chi \in S^h(\Omega_0) \tag{4.41}$$

$$\int_{\Omega_0} (Q - P_N^h Q)\, dx = 0. \tag{4.42}$$

We write

$$u - u_h = (Q - P_N^h Q) + \big[(u-Q) - (u_h - P_N^h Q)\big],$$

and thus,

$$\frac{\partial}{\partial x_i}(u-u_h)(x) = \frac{\partial}{\partial x_i}(Q - P_N^h Q)(x) + r_i(x), \quad \text{for } x \in \Omega_1, \tag{4.43}$$

where

$$\|r_i\|_{L_\infty(\Omega_1)} \le |(u-Q) - (u_h - P_N^h Q)|_{W^1_\infty(\Omega_1)}. \tag{4.44}$$

We will estimate the term on right hand side of this inequality.

2. Since $\mathring{S}^h(\Omega_0) \subset S^h(\Omega_0)$, from (4.3) and (4.41) we have

$$B\big((u-Q) - (u_h - P_N^h Q), \chi\big) = 0, \quad \text{for all } \chi \in \mathring{S}^h(\Omega_0). \tag{4.45}$$

Therefore using (4.44) together with the interior estimate (3.2) with $D = H$, $\Omega_D = \Omega_0$, and Ω_0 in (3.2) replaced by Ω_1 in (4.2), we have

$$\begin{aligned}\|r_i\|_{L_\infty(\Omega_1)} &\le C \min_{\chi \in S^h(\Omega_0)} \Big[\|(u-Q)-\chi\|_{W^1_\infty(\Omega_0)} + H^{-1}\|(u-Q)-\chi\|_{L_\infty(\Omega_0)}\Big] \\ &\quad + CH^{-2}\|(u-Q) - (u_h - P_N^h Q)\|_{L_2(\Omega_0)}. \end{aligned} \tag{4.46}$$

From the approximation axiom (3.3) and (4.40), we get

$$\|(u-Q)-\chi\|_{W^1_\infty(\Omega_0)} \le Ch^k \|u-Q\|_{W^{k+1}_\infty(\tilde{\tilde{\Omega}}_0)} \le Ch^k H.$$

Similarly, we get

$$\|(u-Q)-\chi\|_{L_\infty(\Omega_0)} \le Ch^{k+1}H.$$

Therefore, from (4.46) we have

$$\|r_i\|_{L_\infty(\Omega_1)} \le Ch^k H + CH^{-2}\|(u-Q) - (u_h - P_N^h Q)\|_{L_2(\Omega_0)}, \tag{4.47}$$

where we used the fact that $h < H$.

Now using the assumption (4.18), we get

$$\|u - u_h\|_{L_2(\Omega_0)} \le CH\|u - u_h\|_{L_\infty(\Omega_0)} \le CHh^{k+1-\epsilon}. \tag{4.48}$$

Since Ω_0 is convex, using a standard duality argument, a property of the projection $P_N^h Q$, and (4.40), we get

$$\begin{aligned}\|Q - P_N^h Q\|_{L_2(\Omega_0)} &\le Ch|Q - P_N^h Q|_{H^1(\Omega_0)} \\ &\le Ch^{k+1}\|Q\|_{H^{k+1}(\tilde{\tilde{\Omega}}_0)} \\ &\le Ch^{k+1}\bar{H}\|Q\|_{W_\infty^{k+1}(\tilde{\tilde{\Omega}}_0)} \\ &\le Ch^{k+1}H\|u\|_{W_\infty^{k+1}(\tilde{\tilde{\Omega}}_0)} \le Ch^{k+1-\epsilon}H. \end{aligned} \tag{4.49}$$

Therefore, from (4.47) and (4.48) we have

$$\begin{aligned}\|r_i\|_{L_\infty(\Omega_1)} &\le Ch^k H + CH^{-2}\Big[\|u - u_h\|_{L_2(\Omega_0)} + \|Q - P_N^h Q\|_{L_2(\Omega_0)}\Big] \\ &\le Ch^k H + CH^{-1}h^{k+1-\epsilon}. \end{aligned} \tag{4.50}$$

3. We now consider the term $\frac{\partial}{\partial x_i}(Q - P_N^h Q)$ in (4.43). Let

$$\psi(x) \equiv \rho^h(x) - P_{\text{per}}\rho^h(x),$$

where $\rho^h(x)$ was defined in (4.25) (with Q as in this theorem) and $P_{\text{per}}\rho^h(x)$ is its periodic projection (see (4.30)–(4.31)). We write

$$Q - P_N^h Q = \psi + Q - P_N^h Q - \psi.$$

Thus, for $x \in \Omega_1$,

$$\frac{\partial}{\partial x_i}(Q - P_N^h Q)(x) = \frac{\partial}{\partial x_i}\psi(x) + \bar{r}_i(x), \tag{4.51}$$

where

$$\|\bar{r}_i\|_{L_\infty(\Omega_1)} \le |Q - P_N^h Q - \psi|_{W_\infty^1(\Omega_1)}. \tag{4.52}$$

To estimate the right hand side of the above inequality, we note that

$$\begin{aligned}Q - P_N^h Q - \psi &= Q - P_N^h Q - \rho^h + P_{\text{per}}\rho^h \\ &= Q - P_N^h Q - Q + I^h[Q(\cdot)] + P_{\text{per}}\rho^h \\ &= I^h[Q(\cdot)] - P_N^h Q + P_{\text{per}}\rho^h, \end{aligned}$$

and thus $Q - P_N^h Q - \psi \in S^h(\Omega_0)$.

We recall that $\rho^h \in H^1_{\text{per}}(\Omega_0)$. Therefore from (4.33) and (4.41), we get

$$B(Q - P_N^h Q - \psi, \chi) = 0, \quad \text{for all } \chi \in \mathring{S}^h(\Omega_0).$$

Hence from the interior estimate (3.2) with $u = 0$ (also D, Ω_D, and Ω_0 redefined as before) and using (4.49) and (4.37), we get

$$\begin{aligned} |Q - P_N^h Q - \psi|_{W_\infty^1(\Omega_1)} &\leq CH^{-2}\|Q - P_N^h Q - \psi\|_{L_2(\Omega_0)} \\ &\leq CH^{-2}\|Q - P_N^h Q\|_{L_2(\Omega_0)} + CH^{-2}\|\psi\|_{L_2(\Omega_0)} \\ &\leq CH^{-1}h^{k+1-\epsilon} + CH^{-2}h\|\rho^h\|_{H^1(\Omega_0)}. \end{aligned} \qquad (4.53)$$

Also from (4.24) and (4.40), we have

$$\begin{aligned} \|\rho^h\|_{H^1(\Omega_0)} &= \|Q - I^h[Q(\cdot)]\|_{H^1(\Omega_0)} \\ &\leq Ch^k\|Q\|_{H^{k+1}(\tilde{\tilde{\Omega}}_0)} \\ &\leq Ch^k H\|Q\|_{W_\infty^{k+1}(\tilde{\tilde{\Omega}}_0)} \leq Ch^k H\|u\|_{W_\infty^{k+1}(\tilde{\tilde{\Omega}}_0)} \leq Ch^k H. \end{aligned}$$

Thus from (4.52) and (4.53), we get

$$\|\bar{r}_i\|_{L_\infty(\Omega_1)} \leq |Q - P_N^h Q - \psi|_{W_\infty^1(\Omega_1)} \leq CH^{-1}h^{k+1-\epsilon} + CH^{-1}h^{k+1}. \qquad (4.54)$$

4. Finally combining (4.43), (4.50), (4.51), and (4.54), and writing $R_i(x) = r_i(x) + \bar{r}_i(x)$, we obtain

$$\frac{\partial}{\partial x_i}(u - u_h)(x) = \frac{\partial}{\partial x_i}\psi(x) + R_i(x), \quad \text{for } x \in \Omega_1,$$

where

$$\|R_i\|_{L_\infty(\Omega_1)} \leq \|r_i\|_{L_\infty(\Omega_1)} + \|\bar{r}_i\|_{L_\infty(\Omega_1)} \leq Ch^k H + CH^{-1}h^{k+1-\epsilon} + CH^{-1}h^{k+1}.$$

We now recall that $H = \bar{h}^\beta$, where $\bar{h} = Ch$ and $0 < \beta < 1$ to be determined. Therefore,

$$\|R_i\|_{L_\infty(\Omega_1)} \leq Ch^{k+\beta} + Ch^{k+1-\beta-\epsilon} \leq Ch^k(h^\beta + h^{1-\beta-\epsilon})$$

We choose β such that $0 < \beta < 1 - \epsilon$, and define

$$\alpha = \min(\beta, 1 - \beta - \epsilon),$$

to get

$$\|R_i\|_{L_\infty(\Omega_1)} \leq Ch^{k+\alpha},$$

which is the desired result. □

Remark 4.3 In the proof of Theorem 4.1, we considered $\rho^h(x) = Q(x) - I^h[Q(\cdot)](x)$, where $Q(x)$ was the Taylor polynomial of degree $k+1$ centered at x_0. In fact, it can be easily shown that (4.39) holds when $Q(x)$ is the polynomial $\sum_{i=0}^{k+1}[\partial_1^i\partial_2^{k+1-i}u(0,0)]\, x_1^i x_2^{k+1-i}$ of degree $k+1$ (a linear combination of monomials of degree $(k+1)$).

Remark 4.4 In Remark 4.2, we have seen that the zeros of $\frac{\partial}{\partial x_i}(\rho^h - P_{\text{per}}\rho^h)$ are the superconvergence points. Because of the periodicity of $\rho^h - P_{\text{per}}\rho^h$, we only need the zeros x_0^* of $\frac{\partial}{\partial x_i}(\rho^h - P_{\text{per}}\rho^h)$ in the cell M_0. All other superconvergence points in Ω_1 can be found by a simple translation $x_i^* = x_0^* + x_i$, where $x_0^* + x_i \in \Omega_1$. The points x_0^* can be obtained by finding the zeros $\hat{x}_0^*$ of $\frac{\partial}{\partial \hat{x}_i}(\hat{\rho}^h - \hat{P}_{\text{per}}\hat{\rho}^h)$ in the "master cell" with $\bar{h} = 1$, and then scaling back $\hat{x}_0^*$ to M_0. Thus the computation of x_i^* does not depend on h or the solution u of (2.4).

Remark 4.5 It is immediate from (4.17) and (4.39) that

$$\Big\| \frac{\partial}{\partial x_i}\Big[(u - u_h) - (\rho^h - P_{\text{per}}\rho^h)\Big]\Big\|_{L_\infty(\Omega_1)} \le h^\alpha |u - u_h|_{W^1_\infty(\Omega_1)}$$

and

$$\Big| \frac{\partial}{\partial x_i}(u - u_h)(x_i^*)\Big| \le h^\alpha |u - u_h|_{W^1_\infty(\Omega_1)}$$

where x_i^* is a superconvergence point.

We recall that we considered the partition of unity functions ϕ_i, used in S^{GFEM}, to be C^2 functions, and thus $u_h \in S^{GFEM}$ is also a C^2 function. We will now present a result on the superconvergence related to the second derivatives of $u - u_h$. The analysis will be essentially same as the analysis in Theorem 4.1, but we will require additional assumptions.

Let $\Omega_2 \subset \Omega_1$ be a square centered at x_0 given by

$$\Omega_2 = \big\{x \in \Omega : \|x\|_\infty \le H/2\big\}.$$

In addition to the assumptions in Theorem 4.1, we assume that $k \ge 2$ and

$$Ch^{k-1} \le \|u - u_h\|_{W^2_\infty(\Omega_2)} \tag{4.55}$$

We now present a theorem, which is another important result of this section. In the proof of this theorem, we will use certain technical results obtained in the proof of Theorem 4.1.

Theorem 4.2 *Suppose all the assumptions of Theorem 4.1 hold. Also suppose that (4.55) is satisfied and $k \ge 2$. Let $s = (s_1, s_2)$ be the multi-index with $|s| = 2$. Then, for h small enough, there exists $\alpha > 0$ such that*

$$D^s(u - u_h)(x) = D^s(\rho^h - P_{\text{per}}\rho^h)(x) + R_s(x), \quad x \in \Omega_2, \tag{4.56}$$

where

$$\|R_s\|_{L_\infty(\Omega_2)} \le Ch^{k-1+\alpha} \le Ch^\alpha \|u - u_h\|_{W^2_\infty(\Omega_2)}.$$

and $Q(x)$ in the definition of $\rho^h(x)$ (see (4.25)) is the $(k+1)^{th}$ degree Taylor polynomial of u centered at $x_0 = (0, 0)$.

Remark 4.6 We note that, as in Theorem 4.1, we have also assumed u to be smooth in this theorem ; in particular, u satisfies (4.16) in assumption $\boldsymbol{SC4}$.

Proof: With $Q(x)$ and $P_N^h Q$ as in the proof of Theorem 4.1, we write

$$D^s(u-u_h)(x) = D^s(Q - P_N^h Q)(x) + r_s(x), \quad \text{for } x \in \Omega_2, \tag{4.57}$$

where

$$\|r_s\|_{L_\infty(\Omega_2)} \le \|(u-Q) - (u_h - P_N^h Q)\|_{W^2_\infty(\Omega_2)}. \tag{4.58}$$

Let $v_h \in S^h(\Omega_0)$ be arbitrary. Using the inverse inequality (3.7), we have

$$\begin{aligned}
\|(u-Q) - (u_h - P_N^h Q)\|_{W^2_\infty(\Omega_2)} &\le \|(u-Q) - v_h\|_{W^2_\infty(\Omega_2)} \\
&\quad + \|v_h - (u_h - P_N^h Q)\|_{W^2_\infty(\Omega_2)} \\
&\le \|(u-Q) - v_h\|_{W^2_\infty(\Omega_2)} \\
&\quad + Ch^{-1}\|v_h - (u_h - P_N^h Q)\|_{W^1_\infty(\Omega_1)} \\
&\le \|(u-Q) - v_h\|_{W^2_\infty(\Omega_2)} \\
&\quad + Ch^{-1}\|(u-Q) - v_h\|_{W^1_\infty(\Omega_1)} \\
&\quad + Ch^{-1}\|(u-Q) - (u_h - P_N^h Q)\|_{W^1_\infty(\Omega_1)}.
\end{aligned}$$

Therefore, using the approximation property (3.3) and (4.40) in the proof of Theorem 4.1, we get

$$\begin{aligned}
\|(u-Q) - (u_h - P_N^h Q)\|_{W^2_\infty(\Omega_2)} &\le Ch^{k-1}\|u-Q\|_{W^{k+1}_\infty(\Omega_0)} \\
&\quad + Ch^{-1}\|(u-Q) - (u_h - P_N^h Q)\|_{W^1_\infty(\Omega_1)} \\
&\le CHh^{k-1} \\
&\quad + Ch^{-1}\|(u-Q) - (u_h - P_N^h Q)\|_{W^1_\infty(\Omega_1)}.
\end{aligned} \tag{4.59}$$

A careful examination of the arguments leading to (4.50) shows that

$$\|(u-Q) - (u_h - P_N^h Q)\|_{W^1_\infty(\Omega_1)} \le Ch^k H + CH^{-1}h^{k+1-\epsilon},$$

and thus from (4.58) and (4.59), we get

$$\|r_s\|_{L_\infty(\Omega_2)} \le CHh^{k-1} + CH^{-1}h^{k-\epsilon}. \tag{4.60}$$

Again, with ψ as in the proof of Theorem 4.1, we write

$$D^s(Q - P_N^h Q)(x) = D^s\psi + \bar{r}_s, \quad \text{for } x \in \Omega_2, \tag{4.61}$$

where

$$\|\bar{r}_s\|_{L_\infty(\Omega_2)} \le \|Q - P_N^h Q - \psi\|_{W^2_\infty(\Omega_2)}. \tag{4.62}$$

We have seen in the proof of Theorem 4.1 (in the paragraph after (4.52)) that $Q - P_N^h Q - \psi \in S^h(\Omega_0)$. Therefore using the inverse inequality (3.7) we get,

$$\|Q - P_N^h Q - \psi\|_{W^2_\infty(\Omega_2)} \le Ch^{-1}\|Q - P_N^h Q - \psi\|_{W^1_\infty(\Omega_1)}. \tag{4.63}$$

We have also shown in (4.54) in the proof of Theorem 4.1 that

$$\|Q - P_N^h Q - \psi\|_{W^1_\infty(\Omega_1)} \leq CH^{-1}h^{k+1-\epsilon} + CH^{-1}h^{k+1},$$

and thus from (4.62) and (4.63), we get

$$\|\bar{r}_s\|_{L_\infty(\Omega_2)} \leq CH^{-1}H^{k-\epsilon} + CH^{-1}h^k. \tag{4.64}$$

Finally, writing $R_s(x) = r_s(x) + \bar{r}_s(x)$ and combining (4.57), (4.60), (4.61), and (4.64), we get

$$D^s(u - u_h)(x) = D^s\psi(x) + R_s(x), \quad \text{for } x \in \Omega_2,$$

where

$$\begin{aligned} \|R_s\|_{L_\infty(\Omega_2)} &\leq \|r_s\|_{L_\infty(\Omega_2)} + \|\bar{r}_s\|_{L_\infty(\Omega_2)} \\ &\leq CHh^{k-1} + CH^{-1}h^{k-\epsilon} \\ &= Ch^{k-1}(h^\beta + h^{1-\beta-\epsilon}) \end{aligned}$$

Thus, choosing $0 < \beta < 1 - \epsilon$ and $\alpha = \min(\beta, 1 - \beta - \epsilon)$, and using (4.55), we get

$$\|R_s\|_{L_\infty(\Omega_2)} \leq Ch^{k-1+\alpha} \leq Ch^\alpha \|u - u_h\|_{W^2_\infty(\Omega_2)},$$

which is the desired result. □

Remark 4.7 We note that following the arguments presented in the proof of Theorem 4.2, it is possible to obtain a superconvergence result like (4.56) for higher derivatives of $u - u_h$, i.e., for $D^s(u - u_h)$ for $|s| > 2$. We do not give a proof this result here to keep the exposition simpler.

5 Example

In this section, we will present a computational example to illuminate the results given in Section 4.

We consider the one-dimensional version of the problem (2.1) with $\Omega = (0, 1)$, where the exact solution is $u(x) = \sin(\pi x/2)$. To approximate this solution by GFEM, we choose nodes $x_i = ih$, $i = 0, 1, \cdots, N$, where $Nh = 1$, and define patches ω_i as in Example 1 in Section 2 (see (2.25), (2.24)). For partition of unity functions to be used in the GFEM, we employ functions $\phi_i(x)$, as defined in (2.27), with $r = 0.3$ and $s(x) = q(x - r)$, where

$$q(y) = \left[1 - \left(\frac{y}{1 - 2r}\right)^4\right]^4, \quad 0 \leq y \leq 1 - 2r.$$

We use the space of linear polynomials for local approximating spaces, *i.e.*, $V_j = \mathcal{P}^1(\omega_j)$. We denote the GFEM approximation of $u(x)$ by $u_{GFEM}(x) = u_h(x)$.

It is clear from (4.39) of Theorem 4.1 that, for h small, the superconvergence points of $u' - u'_h$ in $[x_j, x_{j+1}] \subset\subset \Omega$ are the roots of $[\rho^h - P_{\text{per}}\rho^h]'$ in $[x_j, x_{j+1}]$. We obtain these roots by first finding the roots y^* of $\frac{d}{dy}[\hat{\rho} - P_{\text{per}}\hat{\rho}]$ on the master "cell" $0 \leq y \leq 1$. Here

$$\hat{\rho} = y^2 - \sum_{i=0}^{1} \phi_i(y) I_i(y),$$

where $\phi_i(y)$ is the PU function with $h = 1$ and $I_i(y)$ is the Taylor polynomial of y^2, restricted to ω_i with $h = 1$. Also $P_{\text{per}}\hat{\rho}$ is defined as $P_{\text{per}}\hat{\rho} \in S_{\text{per}}$, such that

$$\begin{aligned} \int_0^1 [P_{\text{per}}\hat{\rho}]' v' \, dy &= \int_0^1 \hat{\rho}' v', \quad \text{for all } v \in S_{\text{per}} \\ \int_0^1 P_{\text{per}}\hat{\rho} \, dy &= \int_0^1 \hat{\rho} \, dy \end{aligned}$$

where

$$S_{\text{per}} = \text{span}\{1, \phi_0(y)y + \phi_1(y)(y-1)\}$$

Finally, the superconvergence points x^* of $u' - u'_h$ in $[x_j, x_{j+h}]$ is given by scaling as

$$x^* = x_j + y^* h.$$

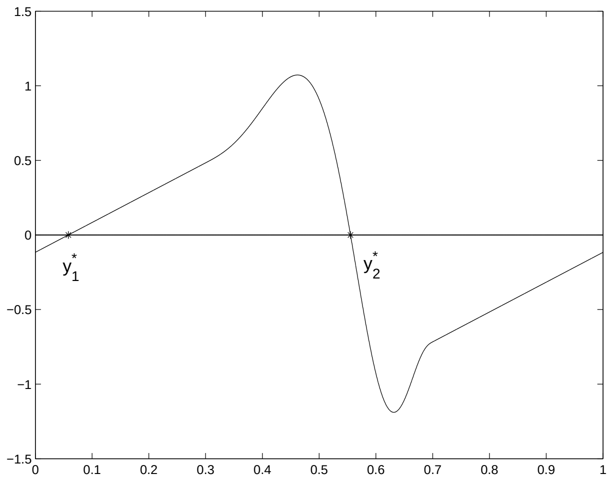
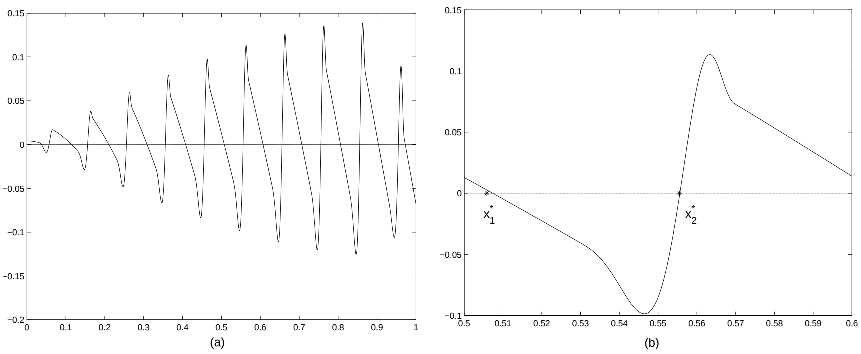


Figure 5.1: Graph of $\frac{d}{dy}[\hat{\rho} - P_{\text{per}}\hat{\rho}]$ on the master cell $[0, 1]$.

In Figure 5.1, we present the graph of $\frac{d}{dy}[\hat{\rho} - P_{\text{per}}\hat{\rho}]$ in $[0, 1]$. The roots of this function are $y_1^* = 0.058309$ and $y_2^* = 0.555216$. Consequently, the superconvergence points of $u' - u'_h$ in $[x_j, x_{j+1}]$ are

$$x_1^* = x_j + 0.058309\, h, \text{ and } x_2^* = x_j + 0.555216\, h. \tag{5.1}$$



Figures 5.2 (a) and (b): (a) Graph of $u' - u_h'$ on $\Omega = (0, 1)$ with $h = 0.1$. (b) Graph of $u' - u_h'$ on $(x_j, x_{j+1}) = (0.5, 0.6)$

In Figure 5.2a, we present the graph of the error $u' - u_h'$ on $\Omega = (0, 1)$, where u_h is the GFEM approximation of u with $h = 0.1$. It is interesting to note that $u' - u_h'$ is zero at several points in the domain $\Omega = (0, 1)$. In Figure 5.2b, we present the graph of $u' - u_h'$ on $(x_j, x_{j+1}) = (0.5, 0.6)$, and also show the superconvergence points $x_1^* = 0.5058309$ and $x_2^* = 0.5555216$. It is clear from Figure 5.2b that $|(u' - u_h')(x_i^*)|$ for $i = 1, 2$ is much smaller than the $\max |(u' - u_h')(x)|$, $.5 \le x \le .6$.

We next computed the GFEM approximation u_h for $h = 0.1, 0.05, 0.025$, and 0.0125. For each value of h, in Table 5.1 we display $M = \max(u' - u_h')(x)$, $x \in [x_j, x_{j+1}] = [0.5, 0.5 + h]$, $e_i' \equiv |(u' - u_h')(x_i^*)|$ and e_i/M for $i = 1, 2$, where x_i^* are the superconvergence points in $[0.5, 0.5 + h]$, given in (5.1).

h	M	e_1'	e_1'/M	e_2'	e_2'/M
0.1	1.13×10^{-1}	2.68×10^{-3}	2.37×10^{-2}	1.89×10^{-3}	1.66×10^{-2}
0.05	5.42×10^{-2}	6.76×10^{-4}	1.25×10^{-2}	4.97×10^{-4}	9.16×10^{-3}
0.025	2.65×10^{-2}	1.70×10^{-4}	6.41×10^{-3}	1.27×10^{-4}	4.80×10^{-3}
0.0125	1.31×10^{-2}	4.30×10^{-5}	3.29×10^{-3}	3.15×10^{-5}	2.41×10^{-3}

Table 5.1

It is clear from Table 5.1 that the ratios e_1'/M and e_2'/M decrease as h decreases, which illuminates the Remark 4.5. It also indicates that x_1^* and x_2^* are indeed superconvergent points of $u' - u_h'$ in $[x_j, x_{j+1}] = [0.5, 0.5 + h]$.

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