Quadrature for Meshless Methods

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Abstract

In this paper we discuss quadrature schemes for meshless methods. We consider the Neumann Problem and derive an estimate for the energy norm error between the exact solution, u, and the quadrature approximate solution, u_h^*, in terms of a parameter, h, associated with the family of approximation spaces, and quantities η, τ, and ϵ that measure the errors in the stiffness matrix, in the lower order term, and in the right-hand side vector, respectively, due to the quadrature. The major hypothesis in the estimate is that the quadrature stiffness matrix has zero row sums, a hypothesis that can be easily achieved by a simple correction of the diagonal elements.

Key Words: Structures, Galerkin methods, Meshfree methods, Quadrature, Error estimates, Row sum condition.

1 Introduction

It has been recognized that one of the major issues concerning Meshless Methods (MM) is the problem of numerical quadrature. In spite of its importance, only a few papers ([1],[2],[3],[4],[5],[6],[7]) address it, and these concentrate mainly on implementational aspects. However, the papers by Chen, *et al* ([2], [3]) recognized clearly that "without doing anything" (*i.e.*, without assuming anything special about the quadrature scheme) the achieved accuracy is very poor. These

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papers proposed a special approach the authors calls *stabilization*, which essentially avoids the problem. The following questions are stimulated by the Chen, *et al* papers:

1. What is the theoretical reason that "doing nothing" leads to poor results?
2. What should one to do to avoid the problem as much as possible?
3. How to characterize theoretically the class of problems for which the approach has desired properties?

The Meshless Methods we consider in this paper are Galerkin methods with the same test and trial functions. There are, of course, other MM, *e.g.*, MM based on collocation (see [6], [8], [9]), which we do not address.

It is worthwhile to mention the differences in the quadrature problem in the classical Finite Element Method (FEM), which was completely analyzed 30 years ago, and in Meshless Methods (MM). The major feature of the FEM is that the shape functions are piecewise polynomials of degree p, and hence their k-th order derivatives vanish on each element for $k \geq p+1$. This permits the exact calculation of the stiffness matrix for PDEs with constant coefficients. With the MM, the shape functions are generally not polynomials, and their k-th order-derivatives grow with k, and essentially no quadrature scheme will be accurate. A second difference is that the *Row Sum Condition* (4.10) for the quadrature stiffness matrix, which is our major hypothesis, is automatically satisfied for the FEM, whereas it is a real condition for MM.

The present paper addresses the quadrature problem for second order elliptic problems and MM of order 1, *i.e.,* when the shape functions reproduce linear functions only. It shows why "doing nothing" leads to nearly catastrophic results. Furthermore, it identifies a simple "recipe" for correcting the elements in the stiffness matrix and the right-hand side vectors so that the inaccuracy problem is eliminated. We analyze the approach on a well-defined class of problems.

The outline of the paper is as follows. Section 2 introduces MMs using non-polynomial approximation spaces associated with uniformly distributed particles. We treat this very special case to motivate the assumptions for the general MM introduced in Section 4. In Section 3 we address the one dimensional problem, showing the effect of using various quadrature schemes which "do nothing", and explaining the character of our results. In Section 4 the MM is introduced as a one-parameter family of methods satisfying several assumptions, presented as axioms. These axioms are discussed especially in light of the specific problem introduced in Section 2. Then in Section 4 our main estimates (Theorems 4.1 – 4.4) for the error, in the presence of quadrature and the correction process for the stiffness matrix and the right-hand side vectors, are stated and proved. Numerical results illustrate these theorems. In this section the Neumann problem for a second order elliptic PDE without a low order term is discussed. Section 5 extends the approach to a general second order PDE with a lower order term. Section 6 presents concluding remarks.

2 Preliminaries

Throughout the paper Ω will be a bounded domain in $\mathbb{R}^d$ with boundary $\Gamma = \partial\Omega$, $H^m(\Omega)$ will be the usual Sobolev space with norm and seminorm, $\|u\|_{m,\Omega}$ and $|u|_{m,\Omega}$, respectively. And $\|u\|_{L_2(\Omega)}, \|u\|_{L_\infty(\Omega)}$, $\|u\|_{L_2(\Gamma)}$, and $\|u\|_{L_\infty(\Gamma)}$ will denote the usual norms on $L_2(\Omega), L_\infty(\Omega)$, $L_2(\Gamma)$, and $L_\infty(\Gamma)$, respectively.

We consider two model boundary value problems,

$$\begin{cases} -\Delta u + \alpha u = f \text{ in } \Omega \\ \frac{\partial u}{\partial n} = g \text{ on } \Gamma = \partial\Omega \end{cases}, \tag{2.1}$$

with $\alpha = 0$ or $\alpha = 1$, where Ω is a bounded domain in $\mathbb{R}^2$ with Liptschitz boundary, $f \in L_2(\Omega)$, and $g \in L_2(\Gamma)$. These two problems have certain essential differences; we discuss them separately in order to clarify the differences and to simplify the analyses.

We first consider the case $\alpha = 0$; in Section 5 we consider the case $\alpha = 1$. In this case ($\alpha = 0$) we assume

$$\int_\Omega f\,dx + \int_\Gamma g\,ds = 0. \tag{2.2}$$

In connection with (2.1), let $H_E = H^1(\Omega)$ denote the Energy Space and $\|u\|_E = |u|_{1,\Omega}$ the Energy Norm. The variational formulation of (2.1) is:

Find $u \in H_E$ satisfying

$$B(u,v) = L(v), \text{ for all } v \in H_E, \tag{2.3}$$

where

$$B(u,v) \equiv \int_\Omega \nabla u \cdot \nabla v\,dx$$

and

$$L(v) \equiv \int_\Omega fv\,dx + \int_\Gamma gv\,ds.$$

Under assumption (2.2), which can be written $L(1) = 0$, the solution to problem (2.1) (or (2.3)) exists and is unique up to an additive constant. We assume in addition to (2.2) that Γ, f, and g are such that u is in $H^2(\Omega)$.

Remark 2.1 We restrict ourselves to purely Neumann Boundary conditions in this paper; other boundary conditions will be treated in a forthcoming paper. The assumption that the coefficients in (2.1) are constant is for simplicity; the general case will be commented on later. The results of the paper are valid for arbitrary dimension d, but in this section we assume $d = 2$ for the sake of simplicity.

We are interested in approximating u by a *Meshless Method* (MM). In this section we consider an example based on uniformly distributed particles, and translation invariant shape functions. Toward this end let

$$x_j^h = (j_1 h, j_2 h) = jh,$$

where $j = (j_1, j_2) \in \mathbb{Z}^2$, with $\mathbb{Z}$ the integer lattice, and $0 < h$, be a family of *uniformly distributed particles.* Suppose $\phi \in H^q(\mathbb{R}^2) \cap L_\infty(\mathbb{R}^2)$, for some $1 \le q$; let $\zeta =$ supp ϕ, and suppose

$$\zeta \subset B_\rho \equiv \{x \in \mathbb{R}^2 : \|x\|_2^2 = x_1^2 + x_2^2 < \rho^2\};$$

ϕ may be non-polynomial. We also suppose $0 \in \mathring{\zeta}$ (= interior of ζ). Then we let

$$\phi_j^h(x) = \phi_j^h(x_1, x_2) \equiv \phi\left(\frac{x - jh}{h}\right) = \phi\left(\frac{x_1 - j_1 h}{h}, \frac{x_2 - j_2 h}{h}\right),$$

for $j \in \mathbb{Z}^2$ and $0 < h$. Clearly

$$\zeta_j^h \equiv \text{ supp } \phi_j^h = \{x : \frac{x - jh}{h} \in \zeta\} \subset B_{\rho h}^j = \{x : \|x - x_j^h\|_2 < \rho h\},$$

and $x_j^h \in \mathring{\zeta}_j^h$. The ϕ_j^h are the *associated particle shape functions.* Particles and particle shape functions defined in this way are *translation invariant*:

$$x_{j+l}^h = x_j^h + x_l^h \text{ and } \phi_{j+l}^h(x) = \phi_j^h(x - x_l^h).$$

We refer to $\phi(x)$ as the *basic shape function.*

We assume that $\{\phi_j^h(x)\}_{j \in \mathbb{Z}^2}$ is *reproducing of order* 1, *i.e.*, that

$$\sum_{j \in \mathbb{Z}^2} (j_1 h)^{i_1} (j_2 h)^{i_2} \phi_j^h(x_1, x_2) = x_1^{i_1} x_2^{i_2}, \text{ for all } x_1, x_2, \tag{2.4}$$

for $0 \le i_1, i_2$ with $i_1 + i_2 \le 1$. We note that $\phi(x)$ can be constructed so that (2.4) is satisfied ([10], [11]). For the remainder of this section, suppose

$$\Omega = \{(x_1, x_2); 0 < x_1 < \pi, 0 < x_2 < \pi\},$$

and suppose $h = \frac{\pi}{n}, n = 1, 2, \cdots$. For each $h = \frac{\pi}{n}$, we consider the family $\phi_j^h(x)$ of shape functions whose supports intersect Ω, and use their restrictions to Ω to define a Galerkin Method for the approximation of the solution u of (2.1) (or (2.3)); this is our MM. Let $\omega_j^h = \mathring{\zeta}_j^h \cap \Omega =$ interior of (supp $\phi_j^h|_\Omega$), and let

$$N_h = \{j : \omega_j^h \ne \emptyset \text{ (the empty set)}\} \text{ and } |N_h| = \text{ cardinality of } N_h.$$

N_h is the set of indices of particles corresponding to shape functions whose supports intersect Ω. Our MM subspace is

$$V_h = \text{span}\{\phi_j^h|_\Omega : j \in N_h\}. \tag{2.5}$$

We will refer to either $\{\phi_j^h : j \in N_h\}$ or $\{\phi_j^h|_\Omega : j \in N_h\}$ as the family of shape functions; this slight abuse of terminology should not cause confusion.

Remark 2.2 As indicated above, we restrict ourselves in this section to uniformly distributed particles and associated shape functions. In Section 4 we will present the theory for the general case, *i.e.,* for general non-uniformly distributed particles.

We assume $\{\phi_j^h|_\Omega : j \in N_h\}$ is linearly independent, and thus a basis for V_h. Our MM approximation, u_h, is defined by

$$\begin{cases} u_h \in V_h \\ B(u_h, v) = L(v), \text{ for all } v \in V_h \end{cases} . \tag{2.6}$$

If we write $u_h(x) = \sum_{j\in N_h} u_{h,j}\phi_j^h(x)$, then u_h satisfies (2.6) if and only if

$$\sum_{j\in N_h} \gamma_{i,j}^h u_{h,j} = l_i^h, \text{ for all } i \in N_h, \tag{2.7}$$

where

$$\gamma_{i,j}^h = B(\phi_i^h, \phi_j^h) = \int_\Omega \nabla\phi_i^h \cdot \nabla\phi_j^h \, dx = \int_{\omega_i^h\cap\omega_j^h} \nabla\phi_i^h \cdot \nabla\phi_j^h \, dx \tag{2.8}$$

and

$$l_i^h = L(\phi_i^h) = \int_\Omega f\phi_i^h \, dx + \int_\Gamma g\phi_i^h \, ds = \int_{\omega_i^h} f\phi_i^h \, dx + \int_{\Gamma\cap\overline{\omega}_i^h} g\phi_i^h \, ds = f_i^h + g_i^h. \tag{2.9}$$

Remark 2.3 . These integrals are computed exactly. Since $\sum_{j\in N_h} \phi_j^h = 1$ (this follows from (2.4)), we see that $\sum_{j\in N_h} \gamma_{i,j}^h = 0$, $\forall i \in N_h$, and that $\sum_{i\in N_h} l_i^h = 0$, which is a necessary and sufficient condition for the solvability of (2.7). We also see that

$$\text{Null space of } \gamma^h = \text{ span } (1, \cdots, 1),$$

where $\gamma^h = \{\gamma_{i,j}^h\}$. Thus, the solution $\overline{u}_h = (u_{h,1}, \cdots, u_{h,|N_h|}) \in \mathbb{R}^{|N_h|}$ of (2.7) exists , and is unique up to an additive constant. (We are using $\overline{u}_h$ to distinguish the vector of coefficients, $(u_{h,1}, \cdots, u_{h,|N_h|})$, from the Galerkin approximation, $u_h = \sum_{j\in N_h} u_{h,j}\phi_j^h$). A unique solution is specified if we impose a constraint of the form $l(\overline{u}_h) = 0$, where l is any linear functional on $\mathbb{R}^{|N_h|}$ satisfying $l(1, \cdots, 1) \neq 0$. If we denote this solution by $\overline{u}_h^l$, then $\overline{u}_h^l + c(1, \cdots, 1)$, with $c \in \mathbb{R}$ arbitrary, yields all the solutions of (2.7). A convenient choice for l is $l(u_{h,1}, \cdots, u_{h,|N_h|}) = u_{h,|N_h|}$. With this choice, we get $\overline{u}_h^l$ by solving (2.7) with the constraint $u_{h,|N_h|} = 0$, which amounts to setting the last coefficient (unknown) equal to 0. We can eliminate one of the equations, say the last, since the rows of γ are linearly dependent and $\sum_{i\in N_h} l_i^h = 0$, obtaining a non-singular square matrix. There are, of course, other choices; *e.g.,* setting $u_{h,1} = 0$ and eliminating one of the equations, but they lead to solutions that differ by constants. All of these solutions, $\overline{u}_h$, of (2.7) lead to Galerkin approximations, u_h, that differ by constants. There are many other possibilities; see [12] for a discussion.

As discussed in the Introduction, the goal of this paper is to understand the effect of using quadrature to evaluate the integrals $\gamma_{i,j}^h = B(\phi_i^h, \phi_j^h)$ and

$l_i^h = L(\phi_i^h)$. Let

$$\gamma_{i,j}^{h*} = ⨍_\Omega \nabla\phi_i^h \cdot \nabla\phi_j^h \, dx = ⨍_{\omega_i^h \cap \omega_j^h} \nabla\phi_i^h \cdot \nabla\phi_j^h \, dx \tag{2.10}$$

and

$$l_i^{h*} = ⨍_\Omega f\phi_i^h \, dx + ⨍_\Gamma g\phi_i^h \, ds = ⨍_{\omega_i^h} f\phi_i^h \, dx + ⨍_{\Gamma\cap\overline{\omega}_i^h} g\phi_i^h \, ds = f_i^{h*} + g_i^{h*}, \tag{2.11}$$

where $⨍$ is a quadrature version of $\int$. Noting that

$$B(u_h, v_h) = \sum_{i,j\in N_h} \gamma_{i,j}^h u_{h,i} v_{h,j} \text{ and } L(v_h) = \sum_{i\in N_h} f_i^h v_{h,i} + \sum_{i\in N_h} g_i^h v_{h,i},$$

for $u_h = \sum_{j\in N_h} u_{h,j}\phi_j^h$ and $v_h = \sum_{j\in N_h} v_{h,j}\phi_j^h$ in V_h, we naturally define

$$B^*(u_h, v_h) = \sum_{i,j\in N_h} \gamma_{i,j}^{h*} u_{h,i} v_{h,j} \text{ and } L^*(v_h) = \sum_{i\in N_h} f_i^{h*} v_{h,i} + \sum_{i\in N_h} g_i^{h*} v_{h,i}. \tag{2.12}$$

The form B^*, the quadrature version of B, is bilinear on $V_h \times V_h$, and the functional L^*, the quadrature version of L, is linear on V_h.

Remark 2.4 Note that $g_i^h = g_i^{h*} = 0$ unless $\overline{\omega}_i^h \cap \Gamma \neq \emptyset$. Setting $N_h' = \{i \in N_h : \overline{\omega}_i^h \cap \Gamma \neq \emptyset\}$, we see that the sum in the second terms in the formulas for $L(v)$ and $L^*(v)$ can be written

$$\sum_{i\in N_h'} g_i^h v_{h,i} \text{ and } \sum_{i\in N_h'} g_i^{h*} v_{h,i};$$

the index i need only be taken over N_h'.

Now define u_h^*, the quadrature approximation to u, by

$$\begin{cases} u_h^* \in V_h \\ B^*(u_h^*, v) = L^*(v), \text{ for all } v \in V_h \end{cases} \tag{2.13}$$

(*cf.* (2.6)). If we write $u_h^* = \sum_{j\in N_h} u_{h,j}^*\phi_j^h(x)$, then u_h^* satisfies (2.13) if and only if

$$\sum_{j\in N_h} \gamma_{i,j}^{h*} u_{h,j}^* = l_i^{h*} = f_i^{h*} + g_i^{h*}, \text{ for all } i \in N_h \tag{2.14}$$

(*cf.* (2.7)).

Remark 2.5 The system (2.14) is quite different than (2.7). There are several situations: (*i*) The system (2.14) is singular with the same structure as (2.7) (*i.e.*, $\sum_{j\in N_h} \gamma_{i,j}^{h*} = 0$ and $\sum_{j\in N_h} l_i^{h*} = 0$); (*ii*) γ^* is singular, but with different structure; and (*iii*) γ^* is nonsingular. Thus (2.14) may have infinitely many solutions, may have no solutions, or may have a unique solution. If we are to base our definition of u_h^* on (2.14), we need some further interpretation of (2.14); see the discussion in Section 3. We further note that when we discuss our main results in Section 4 we will be assuming a hypothesis that ensures that system (2.14) has the same structure as does system (2.7), *i.e.,* we are in situation (i).

Remark 2.6 We are using the same trial and test function, *i.e.,* we are using the Bubnov-Galerkin Method. Formally, we could use different trial and test functions, and obtain the Petrov-Galerkin Method. The analysis of the Petrov-Galerkin Method, however, is subtle even for the usual FEM because stability (the Babŭska-Brezzi Condition ([13], [14])) must be proved. For this reason we do not consider different trial and test functions.

We know that

$$\|u - u_h\|_E \leq Ch\|u\|_{2,\Omega}$$

(see, *e.g.,* [10]). What can be said about

$$\|u - u_h^*\|_E?$$

We will see that $\|u - u_h^*\|_E$ behaves erratically. Later we consider a corrected stiffness matrix, γ^{h**}, and corrected right-hand side vectors, f^{**} and g^{**}, and the corresponding approximate solution, u_h^{**}, and show that u_h^{**} is an accurate approximation to u.

3 A Simple Example/Numerical Results

Consider the example,

$$\begin{cases} -u'' = cos(x),\ x \in \Omega \equiv (0, \pi) \\ u'(0) = u'(\pi) = 0 \end{cases}, \tag{3.1}$$

with variational formulation,

$$\begin{cases} u \in H^1(0, \pi) \\ B(u, v) = F(v),\ \forall v \in H^1(0, \pi), \end{cases}$$

where $B(u, v) = \int_0^\pi u'v'\, dx$ and $F(v) = \int_0^\pi cos\, x\, v\, dx$. This is an example of the one dimensional version of the model problem (2.1) with $\alpha = 0$ and $g = 0$. As stated in Section 2, the solution, u, exists and is unique up to an additive constant.

To construct a meshless method, we let $x_j^h = jh$, for $j \in \mathbb{Z}$, where $h = \pi/n$, for $n = 1, 2, \cdots$, be a family of *uniformly distributed particles.* And we use the *Reproducing Kernel Particle (RKP)* construction, with respect to a *window function*, to construct associated shape functions that are reproducing of order 1. Specifically, we consider the window function

$$w(x) = \exp(\frac{1}{x^2 - r^2}),\ -r < x < r,\ \text{with}\ r = 1.1,$$

with support $[-r, r]$. Using the RKP construction with $h = 1$ (see ([10])), we first calculate $\phi(x)$ with support $[-r, r]$ satisfying

$$\sum_j j^i \phi(x - j) = x^i, \text{ for all } x \text{ and } i = 0, 1.$$

We note that with this construction an explicit formula for $\phi(x)$ is often not available; for each value of $x, \phi(x)$ is determined numerically. The RKP shape functions are then defined by $\phi_j^h(x) = \phi(\frac{x - x_j^h}{h})$; these functions reproduce linear polynomials:

$$\sum_j (x_j^h)^i \phi_j^h(x) = x^i, \text{ for all } x \text{ and } i = 0, 1.$$

Clearly $\zeta_j^h = \text{ supp } \phi_j^h = [x_j^h - hr, x_j^h + hr]$. As in Section 2, we consider the shape functions $\{\phi_j^h(x)\}$ whose supports intersect $\Omega = (0, 1)$, and consider their restrictions to Ω. Let

$$\omega_j^h \equiv (x_j^h - hr, x_j^h + hr) \cap \Omega;$$

then $N_h = \{-1, 0, \ldots, n, n + 1\}$ and $|N_h| = n + 3$.

Remark 3.1 We note that there are many ways, other then RKP, to construct associated shape functions. See [11] , [9].

The MM subspace is

$$V_h = \text{span}\{\phi_j^h(x) : j = -1, 0, \cdots, n, n + 1\} = \{u : u = \sum_{j=-1}^{n+1} u_j \phi_j^h, u_j \in \mathbb{R}\}$$

and the MM is

$$\begin{cases} u \in V_h \\ B(u_h, v) = F(v), \forall v \in V_h. \end{cases} \tag{3.2}$$

The approximate solution, u_h, exists and is unique up to an additive constant. If we write $u_h(x) = \sum_{j=-1}^{n+1} u_{h,j} \phi_j^h(x)$, then u_h satisfies (3.2) if and only if

$$\sum_{j=-1}^{n+1} \gamma_{i,j}^h u_{h,j} = f_i^h, \text{ for } i = -1, \cdots, n + 1, \tag{3.3}$$

where $\gamma_{i,j}^h = B(\phi_i^h, \phi_j^h)$ and $f_i^h = F(\phi_i^h)$.

We then use quadrature to evaluate the integrals $\gamma_{i,j}^h$, but, for the sake of simplicity, the right-hand side is evaluated exactly ($f_i^{h*} = f_i^h$). Let

$$\gamma_{i,j}^{h*} = \int_{\omega_i^h \cap \omega_j^h} (\phi_i^h)' (\phi_j^h)' \, dx,$$

and let B^* be defined as in (2.12). Now define u_h^*, the quadrature approximation to u, by

$$\begin{cases} u_h^* \in V_h \\ B^*(u_h^*, v) = F(v), \text{ for all } v \in V_h \end{cases} \tag{3.4}$$

(*cf.* (2.13)). If we write $u_h^* = \sum_{j=-1}^{n+1} u_{h,j}^* \phi_j^h(x)$, then u_h^* satisfies (3.4) if and only if

$$\sum_{j=-1}^{n+1} \gamma_{i,j}^{h*} u_{h,j}^* = f_i^h, \text{ for } i = -1, \cdots, n+1 \tag{3.5}$$

(*cf.* (2.14)).

As indicated in Remark 2.5, the system (3.5) is quite different than (3.3). It may have infinitely many solutions, may have no solutions, or may have a unique solution. If we are going to use (3.5) to define the approximation with quadrature, we need to have an appropriate interpretation of (3.5). There are various possibilities, for example:

Interpretation 1. We set the last unknown equal to zero, and eliminate the last equation, even though it is not exactly satisfied, and solve the resulting system of $n+2$ equations in $n+2$ unknowns. This system is nonsingular for sufficiently accurate quadrature, and we solve it to get $u_{h,-1}^*, \cdots, u_{h,n}^*$. We then consider $(u_{h,-1}^*, \cdots, u_{h,n}^*, 0)$ to be the solution. Briefly stated, this interpretation is based on the constraint $u_{h,n+1}^* = 0$. We denote the resulting quadrature approximation by u_h^*. This interpretation in used in the one-dimensional ($d = 1$) computations presented below. And it is the interpretation we are using for higher dimension ($d \geq 1$); see Section 2.

Interpretation 2. As suggested in Remark 2.3, there are other possibilities. For example, we could set the first unknown equal to zero, and eliminate one of the other equations, *i.e.,* we could consider the constraint $u_{h,-1}^* = 0$. As seen in Remark 2.3, for the system (3.3), these different possibilities lead to solutions that differ by constants; it is easily seen, however, that this is not true for system (3.5). Thus the different possibilities lead to slightly different definitions of u_h^*.

Remark 3.2 The way we have addressed this issue is similar to the usual approach in the FEM, where the approximate solution is taken to be zero at one point, corresponds to assuming the exact solution is finite and zero at the point. See also [12].

In Figs. 3.1 we present plots of the relative error $\frac{\|u-u_h^*\|_E}{\|u\|_E}$ with respect to h for various quadrature methods: the m-panel Trapezoid Rule; the p-point Gauss Rule; and MATLAB's `quad` (adaptive Simpson quadrature), with tolerance `tol`. Note that different scales for the relative errors are used.

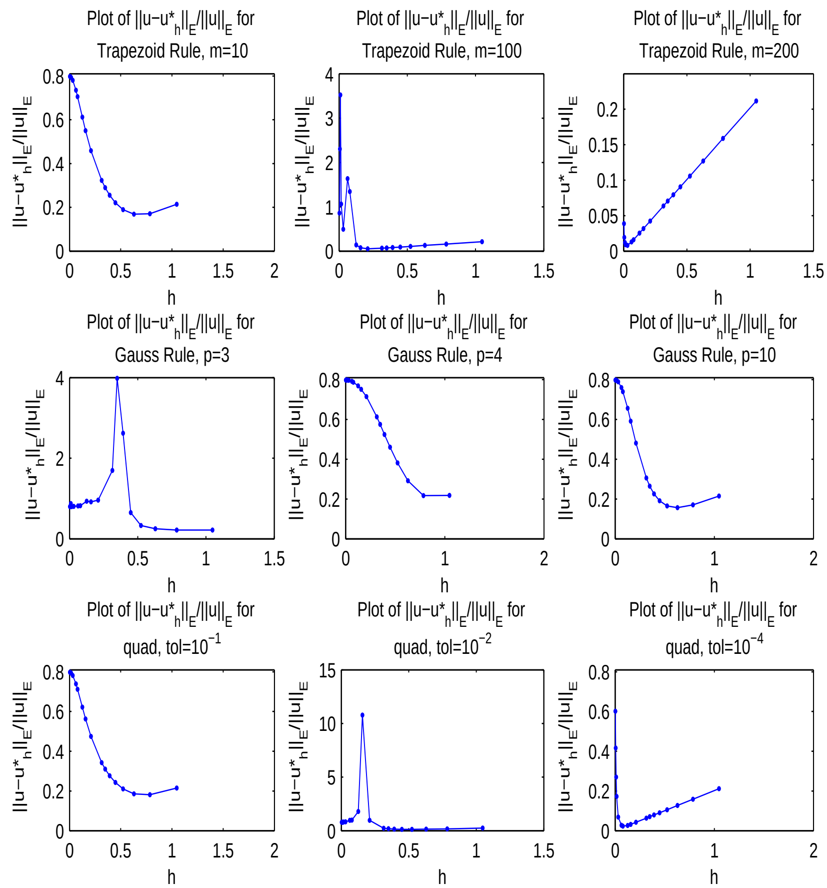

Figure 3.1: The plot of $\frac{\|u-u_h^*\|_E}{\|u\|_E}$ with respect to h for various quadrature schemes. For quadrature schemes we used the m-panel Trapezoid Rule, the p-point Gauss Rule, and MATLAB's `quad` (adaptive Simpson quadrature) with various tolerances, `tol`. Note that different scales for the relative errors are used. We observe that the behavior of the relative error is erratic, and that practically no reasonable accuracy was obtained.

The plots in Fig. 3.1 show that the error is erratic, and that practically no reasonable accuracy was achieved. The error doesn't decrease monotonically as h decreases, and even for higher accuracy (*e.g.,* the 200-panel Trapezoid Rule, the 10-point Gauss Rule, and `quad` with `tol` 10^{-4}), we observed that the error first decreases with decreasing h, but then increases as $h \to 0$. Why did we get such unusual results? How can they be explained? What is the remedy? We note that several of the plots give the impression that the limit of the error as $h \to 0$ is 0.8. But the limiting behavior, as we will see, is more complex.

A partial explanation can be given by carefully examining the following associated periodic problem:

$$-u'' = f(x),\ x \in \mathbb{R}, \tag{3.6}$$

where $f(x)$ is assumed to be $2\pi-$periodic and symmetric at $0, \pm\pi, \cdots$, with $\int_0^\pi f(x)\,dx = 0$, and we seek a solution $u(x)$ that also is 2π-periodic and symmetric at $0, \pm\pi, \cdots$. The solution $u(x)$ exists and is unique to within an additive constant. Let $f(x)$ be the specific function $f(x) = \cos kx$; then

$$u(x) = \frac{1}{k^2}\cos kx + C.$$

Problem (3.6) is essentially the same as the problem (3.1) introduced at the beginning of this section (with the right-hand side replaced by $\cos kx$): if the solution u of (3.1) is reflected symmetrically about 0 and then extended to $\mathbb{R}$ as a 2π-periodic function, the resulting function is the solution (3.6); on the other hand, if the solution to (3.6) is restricted to $[0, \pi]$, the resulting function is the solution to (3.1). Hence we can understand (3.1) by considering (3.6). While the approximation u_h of (3.1) is not the same as the Galerkin approximation of (3.6), they have similar qualitative properties.

First we give the variational formulation of (3.6):

$$\begin{cases} u \in H^1_{\text{per, symm}} \\ B(u, v) = F(v),\ \text{for all } v \in H^1(\mathbb{R}) \text{ with compact support,} \end{cases} \tag{3.7}$$

where

$$H^1_{\text{per, symm}} = \{u \in H^1_{\text{loc}}(\mathbb{R}) : u \text{ is } 2\pi - \text{periodic and symmetric at } 0, \pm\pi, \cdots\}.$$

Since (3.7) is a problem on $\mathbb{R}$, we consider $\phi_j^h(x)$ for all $j \in \mathbb{Z}$, as introduced at the beginning of this section. The Galerkin approximation to u, again denoted by u_h, is then defined by

$$\begin{cases} u_h = \sum_{j\in\mathbb{Z}} u_{h,j}\phi_j^h(x) \in H^1_{\text{per, symm}} \\ B(u_h, \phi_i^h) = \int_{\mathbb{R}} u_h'(\phi_i^h)'\,dx = F(\phi_i^h) = \int_{\mathbb{R}} f\phi_i^h\,dx,\ \text{for all } i \in \mathbb{Z} \end{cases}. \tag{3.8}$$

Since supp $\phi_i^h = \overline{\omega}_i^h$, the integrals in (3.8) are over ω_i^h for each i. The integrals on the left-hand side of (3.8) are computed numerically, but, for the sake of simplicity, we assume the right-hand side is calculated exactly ($f_i^{h*} = f_i^h$). The resulting approximate solution is now denoted by

$$u_h^* = \sum_{j\in\mathbb{Z}} u_{h,j}^*\phi_j^h.$$

It can then be shown that

$$u^*_{h,j} = C\cos kjh, \forall j \in \mathbb{Z},$$

where $C = C_k^{scheme}(h)$ is a constant that depends on k, the quadrature scheme, h, and ϕ. In the case of exact integration we will write "exact" for "scheme" and the approximate solution as u_h. Furthermore, for a particular quadrature scheme, the value of $C = C_k^{scheme}(h)$ can be calculated from the computed stiffness matrix; we do not include this calculation. For exact integration we find that

$$C_1^{exact}(h) = \frac{1 + \mathrm{O}(h^2)}{1 - 0.1039h^2 + \cdots} \tag{3.9}$$

and

$$C_2^{exact}(h) = \frac{1 + \mathrm{O}(h^2)}{4 - 0.1663h^2 + \cdots}. \tag{3.10}$$

(The number 0.1039 appearing in C_1^{exact} is the four digit approximation to the coefficient of h^2; likewise for the other coefficients in these formulas.) As $h \to 0$, $C_1^{exact}(h) \to 1$, which we expect since $u_h \to u$ and $u(x) = \cos x$ when $k = 1$; this is explained by the following calculation:

$$\begin{aligned} u_h(x) = \sum_j u_{h,j}\phi_j^h(x) &= C_1^{exact}(h) \sum_j \cos jh\, \phi_j^h(x) \\ &\approx C_1^{exact}(h)\cos x \approx \cos x = u(x). \end{aligned} \tag{3.11}$$

When $k = 2$, as $h \to 0$, $C_2^{exact}(h) \to 1/4$, which we expect since $u_h \to u$ and $u(x) = \frac{1}{4}\cos 2x$; similar to (3.11) we have

$$u_h^*(x) \approx C_2^{exact}(h)\cos 2x \approx \frac{1}{4}\cos 2x = u(x).$$

Now suppose we are using a quadrature scheme, specifically that we integrate on $\omega_i^h \cap \omega_j^h$ with the Trapezoid Rule (TR) with m-panels. Then, for $m = 10$, we have:

$$C_1^{TR,m=10}(h) = \frac{1 + \mathrm{O}(h^2)}{0.03991h^{-2} + 0.9221 - 0.09753h^2 + \cdots} \tag{3.12}$$

and

$$C_2^{TR,m=10}(h) = \frac{1 + \mathrm{O}(h^2)}{0.03991h^{-2} + 3.6884 - 1.5604h^2 + \cdots}. \tag{3.13}$$

We see that $C_1^{TR,m=10}(h) \to 0$, and hence

$$u_h^*(x) \approx C_1^{TR,m=10}(h)\cos x \approx 0,$$

so the error for small h is $\approx 100\%$ when $k = 1$. Likewise we see that $C_2^{TR,m=10}(h) \to 0$, and hence $u_h^*(x) \to 0$, so the error for small h is $\approx 100\%$ when $k = 2$.

For $m = 100$, we have:

$$C_1^{TR,m=100}(h) = \frac{1 + O(h^2)}{-0.0001975h^{-2} + 1.0000 - 0.1039h^2 + \cdots} \tag{3.14}$$

and

$$C_2^{TR,m=100}(h) = \frac{1 + O(h^2)}{-0.0001975h^{-2} + 4.0000 - 1.6637h^2 + \cdots}. \tag{3.15}$$

For $k = 1$ or 2, we see that for some h, depending on k, $C_k^{TR,m=100}(h)$ is very large, possibly infinite. Hence for $k = 1$ or 2,

$$u_h^*(x) \approx C_k^{TR,m=100}(h) \cos kx$$

is very large, possibly infinite.

For $m = 200$, we have:

$$C_1^{TR,m=200} = \frac{1 + O(h^2)}{0.00000002514h^{-2} + 1.0000 - 0.1039h^2 + \cdots} \tag{3.16}$$

and

$$C_2^{TR,m=200} = \frac{1 + O(h^2)}{0.00000002514h^{-2} + 4.0000 - 1.6637h^2 + \cdots}. \tag{3.17}$$

As when $m = 10$, we see that the solution $u_h^* \to 0$ as $h \to 0$ for $k = 1$ or 2.

In addition, we see from our computations that the coefficient of h^{-2} in the denominators of (3.12)-(3.17) is nonzero. In contrast, the coefficient of h^{-2} in (3.9)-(3.10) is zero. We also observe that in all cases, this coefficient is

$$\textit{Row Sum} = \textit{Row Sum}(i) = \sum_{j \in N_h} \gamma_{i,j}^{h*}.$$

So we can summarize the above observation in terms of Row Sum: If Row Sum is positive, then $u_h^* \to 0$; if Row Sum is negative, then u_h^* is very large, possibly infinite, for certain value of h, *i.e.*, has peaks; briefly u_h is erratic if Row Sum $\neq$ 0.

Remark 3.3 In general Row Sum$(i) = \sum_{j \in N_h} \gamma_{i,j}^{h*}$ depends on i, but for problem (3.8) it is independent of i.

The above analysis shows that the solution of the MM (3.8), with quadrature, behaves erratically when the Row Sum $\neq 0$. So it is reasonable to expect the solution of (3.4), which is an MM with quadrature, will also exhibit erratic behavior, as we have seen in Figure 3.1. In fact, we inferred from (3.14) that the solution of (3.8), with the indicated quadrature, is very large, possibly infinite, for certain values of h. Similar, but not exactly the same, phenomenon can be observed in some of the plots in Figure 3.1, where the error "spikes-up" for certain values of h. We do not, however, claim that the analysis presented here explains all the behaviors seen in Figure 3.1, as the solution of (3.4) is not identical to the solution of (3.8) with quadrature. On the other hand, when Row Sum $= 0$, which is the case with exact integration, the behavior of u_h is as expected. This observation suggests a remedy.

We define

$$\gamma_{i,j}^{h**} = \gamma_{i,j}^{h*}, \text{ for } i \neq j$$

and

$$\gamma_{i,i}^{h**} = - \sum_{j \in N_h, j \neq i} \gamma_{i,j}^{h*}, \tag{3.18}$$

so that

$$\text{Row Sum} = \sum_{j \in N_h} \gamma_{i,j}^{h**} = 0, \, \forall i,$$

and consider the associated approximate solution, u_h^{**}. We call γ^{h**} the *corrected stiffness matrix*. We are correcting the stiffness matrix so as to ensure that Row Sum $= 0$. In Sections 4 and 5, we prove error estimates for $\|u - u_h^{**}\|_E$. We will also consider corrections to the right-hand side vectors f^{h*} and g^{h*}, but we do not consider them here, since we are assuming $f^{h*} = f^h$ and $g = 0$.

Remark 3.4 In a different context, namely, in the computation of pseudospectral differentiation matrices for Čebyšev-Gauss-Lobatto points, an idea similar to (3.18) was used in [15] and in [16]. Specifically, the diagonal elements of the computed differentiation matrices were defined as the negative sums of the off-diagonal elements in the same row. This yielded the desired zero Row Sum for the differentiation matrices.

4 Theory

The MM, as introduced in Section 2, is a Galerkin Method based on an approximation space constructed from uniformly distributed particles and associated shape functions. In this section we consider Galerkin Methods based on approximation spaces that satisfy certain axioms and are of "Meshless Type". By "Meshless Type" we mean approximation spaces spanned by shape functions that do not depend, or depend only minimally, on a mesh. These shape funcions, which are often not polynomials, may be associated with uniformly distributed particles, as in the example in Section 2, or with non-uniformly distributed particles, as in the spaces widely used in engineering, see *e.g.,* Section 4.3 in [10] and the books [8] and [9]. In addition to the axioms on our approximation spaces, we have axioms on the quadrature methods we use.

Let $\Omega \subset \mathbb{R}^d$, and suppose $V_h = V_h(\Omega) \subset H^1(\Omega), 0 < h$, is a one-parameter family of finite dimensional spaces. Suppose $\{\phi_j^h : j \in N_h\}$, where N_h is an index set, is a basis for V_h. Clearly $|N_h| =$ cardinality $of N_h =$ the dimension of V_h, *i.e.,* the number of *Degrees of Freedom.* Let $\omega_j^h =$ interior of supp ϕ_j^h, which is $\subset \Omega$. In this more general context, u_h is defined in (2.6), leading to the linear system (2.7), with $\gamma_{i,j}^h$, f_i^h, and g_i^h defined in (2.8) and (2.9). Using quadrature, we get (2.10)-(2.14). The system (2.14) defines our MM with quadrature in this more general situation. The $\phi_j^h(x)$ are the *shape functions*.

We now state the assumptions we impose on the space V_h and the quadrature method. We formulate them as axioms. Strictly speaking, with this axiomatic approach, we do not have to refer to Meshless Methods; we will, however, take the liberty of referring to Galerkin Methods that satisfy the axioms as Meshless Methods (MM).

- **Axiom 1** The is a constant C, independent of u and h, such that

$$\inf_{\chi \in V_h} \|u - \chi\|_E \leq Ch\|u\|_{2,\Omega}. \tag{4.1}$$

Remark 4.1 Estimate (4.1) is the approximability assumption that would hold in the example in Section 2 provided the shape function $\{\phi_i^h\}$ reproduce polynomials of order 1 (see (2.4)). In this case, it is also clear that the shape functions form a *partition of unity, i.e.,* $\sum_{i \in N_h} \phi_i^h(x) = 1$, $\forall x \in \Omega$. For a discussion of (4.1) for shape functions associated with non-uniformly distributed particles, see ([10]). We mention that the shape functions used in most MMs form a partition of unity, and the associated spaces V_h satisfy (4.1).

- **Axiom 2** There is a constant C, independent of h, such that

$$\sum_{i,j \in N_h} |\gamma_{i,j}^h|(v_i - v_j)^2 \leq C \sum_{i,j \in N_h} (-\gamma_{i,j}^h)(v_i - v_j)^2, \ \forall v = \sum_{i \in N_h} v_i \phi_i^h \in V_h. \tag{4.2}$$

Remark 4.2 Axiom 2 is a fundamental assumption. It is easily seen that it is true for certain finite element approximating spaces. In principle, for any specific family it could be checked numerically. This was done in a couple of cases. Nevertheless the answers to the following questions are not known:

a) What are necessary and sufficient conditions on the V_h for (4.2) to hold?

b) Is the condition (4.2) necessary for our main results.

- **Axiom 3** There is a constant C, independent of h, such that

$$Ch^d \sum_{i \in N_h} v_i^2 \leq \|v\|^2_{L_2(\Omega)} \quad \text{and} \quad Ch^{d-1} \sum_{i \in N_h'} v_i^2 \leq \|v\|^2_{L_2(\Gamma)},$$
$$\forall v \quad = \quad \sum_{i \in N_h} v_i \phi_i^h \in V_h, \tag{4.3}$$

where d is the dimension of the underlying Euclidean space.

Remark 4.3 Axiom 3 holds for the usual hat functions provided the mesh is quasi-uniform; in fact, the two sides of the inequalities are equivalent. The first inequality is related to the linear independence of the shape functions $\{\phi_j^h\}$; it implies linear independence, but is a strengthened uniform version of linear independence. On the other hand, if the translates, $\{\phi(x-j)\}$, of the basic shape function ϕ in Section 2 are linearly independent, then a scaling argument shows that (4.3) holds. Axiom (4.3) also implies that

$$|N_h| \leq Ch^{-d} \text{ and } |N_h'| \leq Ch^{1-d} \tag{4.4}$$

provided the ϕ_j^h form a partition of unity. We will see in Remark 4.11 that Axioms 3, (4.9), and (4.10) imply that $\{\phi_i^h\}$ forms a partition of unity.

There are certain situations in which (4.3) can be imposed by appropriately selecting the location of the particles x_j^h. We explain this in terms of a particular example. Let x_j^h and ϕ_j^h be as in Section 2, but suppose Ω is a circle (instead of a square). Also suppose that if $\mathring{\zeta}_j^h \subset \Omega$, then $|\omega_j^h| = |\mathring{\zeta}_j^h| = O(h^d)$. Then for particles near the boundary of $\Omega, |\omega_j^h| = |\mathring{\zeta}_j^h \cap \Omega|$ may be much smaller than h^d, and (4.3) may not hold ([17]). However, by appropriately modifying the locations of some of the particles near the boundary, (4.3) can be made to be valid.

- **Axiom 4** For $i \in N_h$, there is a finite number m_i of indices j such that $\omega_i^h \cap \omega_j^h \neq \emptyset$, and there is a constant κ, independent of i, j and h, such that

$$m_i \leq \kappa. \tag{4.5}$$

Remark 4.4 This condition is imposed in the construction of most MMs. In the example discussed in Section 2, (4.5) holds with κ depending on ρ.

- **Axiom 5** There are constants ν and C, independent of i and h, such that

$$\|\phi_i^h\|_{L_\infty(\Omega)} \leq \nu, \ |\omega_i^h| \leq Ch^d, \text{ and } |\overline{\omega}_i^h \cap \Gamma| \leq Ch^{d-1}, \tag{4.6}$$

where $|\omega_i^h|$ = "area" of ω_i^h in $\mathbb{R}^d$ and $|\overline{\omega}_i^h \cap \Gamma|$ = "length" of ω_i^h in $\mathbb{R}^{d-1}$.

Remark 4.5 In the example discussed in Section 2, the first inequality in (4.6) holds with $\nu = \|\phi\|_{L_\infty(\mathbb{R}^d)}$. The 2nd and 3rd hold with C depending on ρ. Axioms (4.3) and (4.6) show that the two sides the inequalities in (4.3) are equivalent. Furthermore, $|N_h| \geq Ch^{-d}$ and $|N_h'| \geq Ch^{1-d}$, and hence

$$\frac{|N_h'|}{|N_h|} \leq rh \text{ and } \frac{|N_h|}{|N_h'|} \leq \tilde{r}h^{-1}, \tag{4.7}$$

where r and $\tilde{r}$ are independent of h.

- **Axiom 6** There is a constant C, independent of i, j, and h, such that

$$\text{for each } i,\ |\gamma_{i,j}^h| \le C|\gamma_{i,i}^h|,\ \forall j, h. \tag{4.8}$$

Remark 4.6 Axiom 6 is a reasonable assumption since $\gamma_{i,j}^h$ is an integral of a positive function over ω_i^h, and $\gamma_{i,j}^h$ is an integral over $\omega_i^h \cap \omega_j^h$. So, if $|\gamma_{i,i}^h|$ is small by virtue of the particle x_i^h being near the boundary of Ω, then the quantities $|\gamma_{i,j}^h|, \forall j$ are also small.

- **Axiom 7**

$$\gamma_{i,j}^h = \gamma_{j,i}^h \text{ and } \gamma_{i,j}^{h*} = \gamma_{j,i}^{h*},\ \forall i, j. \tag{4.9}$$

Remark 4.7 Symmetry is, of course, trivially true for both $\gamma^h = \{\gamma_{i,j}^h\}$, the stiffness matrix with exact integration, and $\gamma^{h*} = \{\gamma_{i,j}^{h*}\}$, the quadrature stiffness matrix.

- **Axiom 8**

$$\sum_{j \in N_h} \gamma_{i,j}^h = 0 \text{ and } \sum_{j \in N_h} \gamma_{i,j}^{h*} = 0,\ \forall i. \tag{4.10}$$

Remark 4.8 The equations in (4.10) are called the *Row Sum Condition* for γ^h and γ^{h*}, respectively. The Row Sum Condition for γ^h holds provided that $\sum_{i \in N_h} \phi_i^h(x) = c$, $\forall x$, with c a constant (which is the notion of partition of unity when $c = 1$), as seen from the calculation,

$$\begin{aligned} \sum_{j \in N_h} \gamma_{i,j}^h = \sum_{j \in N_h} \int_\Omega \nabla\phi_i^h \cdot \nabla\phi_j^h \, dx &= \int_\Omega \nabla\phi_i^h \cdot \nabla(\sum_{j \in N_h} \phi_j^h) \, dx \\ &= \int_\Omega \nabla\phi_i^h \cdot \nabla c \, dx \, dy = 0. \end{aligned}$$

We will see in Remark 4.11 that the Row Sum Condition for γ^h implies that $\sum_{i \in N_h} \phi_i^h = c$, so that the two conditions are equivalent. The Row Sum Condition for γ^h implies that the linear system (2.7) is singular.

It is informative to try to mimic the above calculation for γ^{h*}:

$$\begin{aligned} \sum_{j \in N_h} \gamma_{i,j}^{h*} = \sum_{j} ⨍_\Omega \nabla\phi_i^h \cdot \nabla\phi_j^h \, dx &= ⨍_\Omega \nabla\phi_i^h \cdot \nabla(\sum_{j \in N_h} \phi_j^h) \, dx \\ &= ⨍_\Omega \nabla\phi_i^h \cdot \nabla c \, dx \, dy = 0, \end{aligned}$$

provided $\sum_{i \in N_h} \phi_i^h(x) = c$ and provided $⨍$ is linear, specifically is additive in its second argument. If $⨍$ is formed with a quadrature scheme over Ω, then $⨍$ would be linear. But, noting that

$$\int_\Omega \nabla\phi_i^h \cdot \nabla\phi_j^h \, dxdy = \int_{\omega_i^h \cap \omega_j^h} \nabla\phi_i^h \cdot \nabla\phi_j^h \, dxdy,$$

we see that a quadrature scheme for $\int_\Omega \nabla\phi_i^h \cdot \nabla\phi_j^h \, dxdy$ can be based on $\omega_i^h \cap \omega_j^h$ or a box containing $\omega_i^h \cap \omega_j^h$, as well as on Ω. With these first two choices, $⨍$ would not be linear: we would be using a quadrature scheme that depended on the integrand, $\nabla\phi_i^h \cdot \nabla\phi_j^h$. In this situation,

$$\sum_{j\in N_h} \gamma_{i,j}^{h*} = 0, \forall i$$

is indeed a real condition. It implies that the linear system (2.14) is singular.

Note that in the FEM the quadrature schemes are triangle based (a separate quadrature scheme on each triangle in the triangulation, $\{\tau_h\}$, of the domain Ω), and viewing integrals as over Ω as we compute

$$\gamma_{i,j}^h = ⨍_\Omega \nabla\phi_i^h \cdot \nabla\phi_j^h \, dxdy = \sum_{T\in\tau_h} ⨍_T \nabla\phi_i^h \cdot \nabla\phi_j^h \, dxdy.$$

It is immediate the $⨍_\Omega$ is linear, so the Row Sum Condition is satisfied for FEM. In contrast, in MM it is the absence of the decomposition of Ω into triangles, and the use of quadrature schemes based on the intersections $\omega_i^h \cap \omega_j^h$ that leads to $⨍$ that is not linear.

We further note that the Row Sum Condition for γ^* is equivalent to the following condition:

For any constant solution u of (2.1), we have $u_h^* =$ a constant.

Note: If u is a constant solution, then $f = g = 0$.

- **Axiom 9**

$$\sum_{i\in N_h} (f_i^h + g_i^h) = \sum_{i\in N_h} f_i^h + \sum_{i\in N_h'} g_i^h = 0,$$
$$\sum_{i\in N_h} (f_i^{h*} + g_i^{h*}) = \sum_{i\in N_h} f_i^{h*} + \sum_{i\in N_h'} g_i^{h*} = 0. \tag{4.11}$$

Remark 4.9 The first equation in (4.11) follows directly from $\int_\Omega f\, dx + \int_\Gamma g\, ds = 0$ provided that $\{\phi_i^h\}$ forms a partition of unity. It is the compatibility condition implying the system (2.7) is solvable. The second equation is a real condition. It is the compatibility condition showing the the singular system (2.14) is solvable. The equations in (4.11) are called the *Right-Hand Side Sum Conditions*.

- **Axiom 10**

$$\gamma_{i,j}^{h*} = \gamma_{i,j}^h + \eta_{i,j}^h,\ f_i^{h*} = f_i^h + \epsilon_i^h, \text{ and } g_i^{h*} = g_i^h + \tau_i^h, \tag{4.12}$$

with

$$|\eta^h_{i,j}| \le \eta \max(|\gamma^h_{i,j}|, \nu h^d),$$
$$|\epsilon^h_i| \le \epsilon \max(|f^h_i|, \nu h^d \|f\|_{L_\infty(\Omega)}), \text{ and}$$
$$|\tau^h_i| \le \tau \max(|g^h_i|, \nu h^{d-1} \|g\|_{L_\infty(\Gamma)}).$$

Remark 4.10 Axiom 10 is based on the typical accuracy estimate when adaptive integration with tolerances η, ϵ, and τ are used. Note that the size of $\gamma^h_{i,j}$ is bounded above by Ch^{d-2}, f^h_i by Ch^d, and g^h_i by Ch^{d-1}. In the implementation, we compute both the relative and absolute errors, and stop the computation when either the relative or absolute error is admissible. Usually the relative error governs the termination, especially for $\gamma^h_{i,j}$.

Here are some consequences of Axiom 10:

Let

$$K = \{(i,j) : i, j \in N_h\} \text{ and } K_0 = \{(i,j) : \omega^h_i \cap \omega^h_j = \emptyset\}.$$

Then let

$$K_1 = \{(i,j) : (i,j) \notin K_0, |\gamma^h_{i,j}| \ge \nu h^d\}$$

and

$$K_2 = \{(i,j) : (i,j) \notin K_0, |\gamma^h_{i,j}| < \nu h^d\}.$$

With these definitions, K_0, K_1, K_2 are pairwise disjoint, $K = K_0 \cup K_1 \cup K_2$,

$$\gamma^h_{i,j} = \gamma^{h**}_{i,j} = \eta^h_{i,j} = 0, \, \forall (i,j) \in K_0,$$

and

$$|\eta^h_{i,j}| \le \eta |\gamma^h_{i,j}|, \, \forall (i,j) \in K_1 \text{ and } |\eta^h_{i,j}| \le \eta \nu h^d, \, \forall (i,j) \in K_2. \tag{4.13}$$

Since

$$|f^h_i| = |\int_\Omega f \phi^h_i \, dx| \le \|f\|_{L_\infty(\Omega)} \nu |\omega^h_i| \le C_1 \nu \|f\|_{L_\infty(\Omega)} h^d, \tag{4.14}$$

where C_1 is the constant in Axiom 5, from Axiom (4.12) we have

$$|f^{h*}_i - f^h_i| = |\epsilon^h_i| \le \max(C_1, 1) \epsilon \nu h^d \|f\|_{L_\infty(\Omega)}, \, \forall i; \tag{4.15}$$

and, since

$$|g^h_i| = |\int_{\Gamma \cap \overline{\omega}^h_i} g \phi^h_i \, ds| \le \|g\|_{L_\infty(\Gamma)} \nu |\Gamma \cap \omega^h_i| \le C_1 \nu \|g\|_{L_\infty(\Gamma)} h^{d-1}, \tag{4.16}$$

from Axiom (4.12) we have

$$|g^{h*}_i - g^h_i| = |\tau^h_i| \le \max(C_1, 1) \tau \nu h^{d-1} \|g\|_{L_2(\Gamma)}, \, \forall i. \tag{4.17}$$

A family $\phi_j^h|_\Omega, j \in N_h$, satisfying Axions 1–10 will be called a MM family of shape functions. It is immediate that the family $\phi_j^h(x)$ of particle shape functions in Section 2 is a MM family of shape functions provided they satisfy (4.2) and (4.3). A second example is provided by the non-uniformly distributed RKP shape functions widely used in engineering. Strictly speaking, as pointed out above, when considering Galerkin Methods with approximation spaces satisfying the above axioms, we do not have to refer to Meshless Methods. It is, however, important to note that important specific MM fit within our framework, and so our results (Theorems 4.1 – 4.4, 5.1 – 5.4) apply to them.

One more point: In usual treatments of Galerkin Methods, the major assumption involves approximability, as in our Axiom 1. We have several additional axioms. What is their purpose? They make possible the analysis of quadrature.

We now turn to the proof of our results. We begin by proving four preliminary lemmas.

Lemma 4.1

$$-2B(w,v) = \sum_{i,j\in N_h} \gamma_{i,j}^h (w_i - w_j)(v_i - v_j),\ \forall w = \sum_{i\in N_h} w_i\phi_i^h, v = \sum_{i\in N_h} v_i\phi_i^h \in V_h. \tag{4.18}$$

Proof. Using the symmetry of γ^h and the Row Sum Condition for γ^h, *i.e.,* the 1st equation in Axiom 8, $\forall w = \sum_{i\in N_h} w_i\phi_i^h, v = \sum_{i\in N_h} v_i\phi_i^h \in V_h$, we have

$$\begin{aligned}
\sum_{i,j\in N_h} \gamma_{i,j}^h (w_i - w_j)(v_i - v_j) &= \sum_{i,j\in N_h} \gamma_{i,j}^h w_i v_i + \sum_{i,j\in N_h} \gamma_{i,j}^h w_j v_j \\
&\quad - \sum_{i,j\in N_h} \gamma_{i,j}^h w_i v_j - \sum_{i,j\in N_h} \gamma_{i,j}^h w_j v_i \\
&= \sum_{i\in N_h} w_i v_i \sum_{j\in N_h} \gamma_{i,j}^h + \sum_{j\in N_h} w_j v_j \sum_{i\in N_h} \gamma_{i,j}^h \\
&\quad - \sum_{i,j\in N_h} \gamma_{i,j}^h w_i v_j - \sum_{i,j\in N_h} \gamma_{j,i}^h w_i v_j \\
&= -2 \sum_{i,j\in N_h} \gamma_{i,j}^h w_i v_j \\
&= -2B(w,v).
\end{aligned}$$

□

Remark 4.11 In Remark 4.8 we showed that the Row Sum Condition for γ^h follows from $\sum_{i\in N_h} \phi_i^h(x) = c$. The converse is also true. To see this, suppose that the Row Sum Condition is satisfied, and that γ^h is symmetric (*cf.* (4.9)). Then we have formula (4.18). In this formula let $v = w$ with $w_i = c$. Then

$$-2B(w,w) = \sum_{i,j\in N_h} \gamma_{i,j}^h (w_i - w_j)^2 = 0,$$

which implies that

$$w = \sum_{i \in N_h} w_i \phi_i^h = c \sum_{i \in N_h} \phi_i^h = c = \text{ constant},$$

as desired.

Lemma 4.2

$$-2B^*(w,v) = \sum_{i,j \in N_h} \gamma_{i,j}^{h*}(w_i - w_j)(v_i - v_j),\ \forall w = \sum_{i \in N_h} w_i \phi_i^h, v = \sum_{i \in N_h} v_i \phi_h^i \in V_h. \tag{4.19}$$

Proof. Recalling that $B^*(w,v) \equiv \sum_{i,j \in N_h} \gamma_{i,j}^{h*} w_i v_j$, the proof is similar to the proof of Lemma 4.1, but uses the symmetry of γ^{h*} (*cf.* (4.9)) and the 2nd equation in Axiom 8. □

Lemma 4.3 *There is a constant C, independent of w, v and h, such that*

$$|B^*(w,v)| \leq [1 + C\eta]\, \|w\|_E \|v\|_E,\ \forall w, v \in V_h, \tag{4.20}$$

where η is the error parameter in Axiom 10.

Proof. From Lemmas 4.1 and 4.2, Axiom 10, and Remark 4.10 we have

$$\begin{aligned} B^*(w,v) &= -\frac{1}{2} \sum_{(i,j) \in K} \gamma_{i,j}^{h*}(w_i - w_j)(v_i - v_j) \\ &= -\frac{1}{2} \sum_{(i,j) \in K} \gamma_{i,j}^{h}(w_i - w_j)(v_i - v_j) \\ &\qquad -\frac{1}{2} \sum_{(i,j) \in K} \eta_{i,j}^{h}(w_i - w_j)(v_i - v_j) \\ &= B(w,v) - \frac{1}{2} \sum_{(i,j) \in K_1} \eta_{i,j}^{h}(w_i - w_j)(v_i - v_j) \\ &\qquad -\frac{1}{2} \sum_{(i,j) \in K_2} \eta_{i,j}^{h}(w_i - w_j)(v_i - v_j), \end{aligned}$$

for $w = \sum_i w_i \phi_i^h, v = \sum_i v_i \phi_i^h \in V_h$.

Suppose C_2 is the constant in Axiom 2 and ν is the constant in Axiom 5. Then, using the above formula, Remark 4.10 again, the Schwartz inequality, Axiom 2, and Lemma 4.1, we obtain

$$\begin{aligned}
|B^*(w,v)| &\leq |B(w,v)| \\
&\quad + \frac{\eta}{2}\left(\sum_{(i,j)\in K_1} |\gamma_{i,j}^h|(w_i - w_j)^2\right)^{1/2}\left(\sum_{(i,j)\in K_1} |\gamma_{i,j}^h|(v_i - v_j)^2\right)^{1/2} \\
&\quad + \frac{\eta\nu h^d}{2}\left(\sum_{(i,j)\in K_2} (w_i - w_j)^2\right)^{1/2}\left(\sum_{(i,j)\in K_2} (v_i - v_j)^2\right)^{1/2} \\
&\leq |B(w,v)| \qquad + \frac{C_2\eta}{2}\left(\sum_{(i,j)\in K} (-\gamma_{i,j}^h)(w_i - w_j)^2\right)^{1/2} \\
&\qquad \times \left(\sum_{(i,j)\in K} (-\gamma_{i,j}^h)(v_i - v_j)^2\right)^{1/2} \\
&\qquad + \eta\nu h^d\left[\left(\sum_{(i,j)\in K_2} w_i^2\right)^{1/2} + \left(\sum_{(i,j)\in K_2} w_j^2\right)^{1/2}\right] \\
&\qquad \times \left[\left(\sum_{(i,j)\in K_2} v_i^2\right)^{1/2} + \left(\sum_{(i,j)\in K_2} v_j^2\right)^{1/2}\right] \\
&\leq |B(w,v)| + C_2\eta B(w,w)^{1/2}B(v,v)^{1/2} \\
&\quad + \eta\nu h^d\left[\left(\sum_{(i,j)\in K_2} w_i^2\right)^{1/2} + \left(\sum_{(i,j)\in K_2} w_j^2\right)^{1/2}\right] \\
&\quad \times \left[\left(\sum_{(i,j)\in K_2} v_i^2\right)^{1/2} + \left(\sum_{(i,j)\in K_2} v_j^2\right)^{1/2}\right].
\end{aligned} \tag{4.21}$$

Now, using Axioms 3 and 4 we see that

$$\sum_{(i,j)\in K_2} w_i^2 \leq \sum_{i\in N_h}\sum_{j:(i,j)\in K_2} w_i^2 \leq \kappa \sum_{i\in N_h} w_i^2 \leq \kappa C_3^{-1} h^{-d}\|w\|_{L_2}^2, \tag{4.22}$$

where κ is the constant in Axiom 4 and C_3 is the constant in Axiom 3; likewise the quantities $\sum_{(i,j)\in K_2} w_j^2, \sum_{(i,j)\in K_2} v_i^2$, and $\sum_{(i,j)\in K_2} v_j^2$ are bounded by $\kappa C_3^{-1}h^{-d}\|w\|_{L_2(\Omega)}^2$ and $\kappa C_3^{-1}h^{-d}\|v\|_{L_2(\Omega)}^2$, respectively. Using these estimates in (4.21) yields

$$|B^*(w,v)| \leq |B(w,v)| + C_2\eta B(w,w)^{1/2}B(v,v)^{1/2} + 4\eta\nu\kappa C_3^{-1}\|w\|_{L_2(\Omega)}\|v\|_{L_2(\Omega)}.$$

Apply these estimates to $\overline{w} = w - \frac{\int_\Omega w\,dx}{|\Omega|}$ and $\overline{v} = v - \frac{\int_\Omega v\,dx}{|\Omega|}$. Since $B^*(\overline{w}, \overline{v}) = B^*(w, v)$ (apply Lemma 4.1), $B(\overline{w}, \overline{v}) = B(w, v)$, and $\overline{w}$ and $\overline{v}$ have average 0, we can use the Poincaré inequality for functions with zero average to obtain

$$|B^*(w,v)| \leq \left[1 + \eta(C_2 + 4\nu\kappa C_3^{-1} C_4)\right] \|w\|_E \|v\|_E,$$

where C_4 is the constant in the Poincaré inequality. This is estimate (4.20) with $C = (C_2 + 4\nu\kappa C_3^{-1} C_4)$.
□

Lemma 4.4 *There is a constant C, independent of $\eta, w,$ and h, such that*

$$B^*(w,w) \geq [1 - C\eta] \|w\|_E^2, \forall w \in V_h \textit{ with zero average.} \tag{4.23}$$

Proof. Suppose

$$|\eta| < \frac{2}{C_2 + 4\kappa\nu C_3^{-1} C_4},$$

where $\eta, \nu, \kappa, C_2, C_3,$ and C_4 are as in Lemma 4.3. Then, using Lemmas 4.1 and 4.2, Axioms 10 and 2, Remark 4.10, and the Poincaré inequality, and estimate (4.22), for any $w = \sum_{i \in N_h} w_i \phi_i^h$ with zero average, we have

$$\begin{aligned}
B^*(w,w) &= \frac{1}{2} \sum_{(i,j)\in K} (-\gamma_{i,j}^{h*})(w_i - w_j)^2 \\
&= B(w,w) - \frac{1}{2} \sum_{(i,j)\in K} \eta_{i,j}^h (w_i - w_j)^2 \\
&\geq B(w,w) - \frac{\eta}{2} \sum_{(i,j)\in K_1} |\gamma_{i,j}^h| (w_i - w_j)^2 - \frac{\eta\nu h^d}{2} \sum_{(i,j)\in K_2} (w_i - w_j)^2 \\
&\geq B(w,w) - \frac{\eta C_2}{2} \sum_{(i,j)\in K_1} (-\gamma_{i,j}^h)(w_i - w_j)^2 \\
&\qquad - \frac{\eta\nu h^d}{2} \sum_{(i,j)\in K_2} (w_i - w_j)^2 \\
&\geq B(w,w) - \frac{\eta C_2}{2} B(w,w) - \eta\nu h^d \left(\sum_{(i,j)\in K_2} w_i^2 + \sum_{(i,j)\in K_2} w_j^2 \right) \\
&\geq B(w,w) - \frac{\eta C_2}{2} B(w,w) - 2\eta\nu\kappa C_3^{-1} \|w\|_{L_2(\Omega)}^2 \\
&\geq \left[1 - \eta\left(\frac{C_2}{2} + 2\kappa\nu C_3^{-1} C_4\right)\right] \|w\|_E^2,
\end{aligned}$$

which is (4.23) with $C = \frac{C_2}{2} + 2\kappa\nu C_3^{-1} C_4$. □

Remark 4.12 Lemma 4.3 establishes the boundedness of B^*, uniformly in h, and Lemma 4.4 establishes the coercivity of B^*, uniformly in h.

Remark 4.13 In Lemmas 4.3 and 4.4 we have used explicit constants (C_1, C_2, *etc.*) because they provide useful information. In the remainder the paper, we will usually use generic constants.

We now state and prove an estimate for $\|u_h^* - u_h\|_E$.

Theorem 4.1 *Suppose our shape functions and our quadrature scheme satisfy Axioms 1–10. Then for small* η*, there is a constant* C*, independent of* $u, \epsilon, \tau, \eta,$ *and* h*, such that*

$$\|u_h - u_h^*\|_E \leq C\left[\epsilon\|f\|_{L_\infty(\Omega)} + \tau\|g\|_{L_\infty(\Gamma)} + \eta\|u\|_E\right], \forall h. \tag{4.24}$$

Proof. The exact solution u is characterized by (*cf.* (2.3)):

$$\begin{cases} u \in H^1(\Omega) \\ B(u,v) = L(v), \forall v \in H^1(\Omega); \end{cases}$$

the exact integration Galerkin approximation u_h of u, is characterized by (*cf.* (2.6)):

$$\begin{cases} u_h \in V_h = \text{ span } \{\phi_j\} \\ B(u_h,v) = L(v), \forall v \in V_h; \end{cases}$$

and the quadrature approximate solution u_h^*, is characterized by (*cf.* (2.13)):

$$\begin{cases} u_h^* \in V_h \\ B^*(u_h^*,v) = L^*(v), \forall v \in V_h. \end{cases}$$

As pointed out in Section 2, u and u_h exist and are unique up to additive constants. Because of Assumptions (4.10) and (4.11), u_h^* also exists and is unique up to an additive constant (*cf.* Remarks 2.3 and 2.5). Since the Energy Norm is zero on constants, the non-uniqueness in u_h and u_h^* doesn't affect the estimate (4.24).

Using Lemmas 4.1 and 4.2 and Axiom 10 with $w = u_h = \sum_{i\in N_h} u_{h,i}\phi_i^h$, we get

$$\begin{aligned} -2B^*(u_h,v) &= \sum_{(i,j)\in K} \gamma_{i,j}^{h*}(u_{h,i}-u_{h,j})(v_i-v_j) \\ &= \sum_{(i,j)\in K} \gamma_{i,j}^{h}(u_{h,i}-u_{h,j})(v_i-v_j) \\ &\quad + \sum_{(i,j)\in K} \eta_{i,j}^{h}(u_{h,i}-u_{h,j})(v_i-v_j) \\ &= -2B(u_h,v) + Q(u_h,v), \forall v = \sum_{i\in N_h} v_i\phi_i^h \in V_h, \end{aligned}$$

where $Q(u_h,v) \equiv \sum_{(i,j)\in K} \eta^h_{i,j}(u_{h,i}-u_{h,j})(v_i-v_j)$. Then, using the characterizations of u_h and u_h^*, we obtain

$$\begin{aligned}
B^*(u_h^*-u_h,v) &= B^*(u_h^*,v)-B^*(u_h,v)\\
&= B^*(u_h^*,v)-B(u_h,v)+\frac{1}{2}Q(u_h,v)\\
&= L^*(v)-L(v)+\frac{1}{2}Q(u_h,v),\ \forall v\in V_h. \qquad (4.25)
\end{aligned}$$

Now we estimate $Q(u_h,v)$ and $L^*(v)-L(v)$. We first observe that we can assume that $\int_\Omega u_h\,dx = 0$. From Lemmas 4.1 and 4.2, Axioms 10 and 2, estimate (4.22), Remark 4.10, and the Poincaré inequality, we have

$$\begin{aligned}
|Q(u_h,v)| &= \Big|\sum_{(i,j)\in K_1} \eta^h_{i,j}(u_{h,j}-u_{h,i})(v_i-v_j)\\
&\qquad + \sum_{(i,j)\in K_2} \eta^h_{i,j}(u_{h,j}-u_{h,i})(v_i-v_j)\Big|\\
&\le \eta \sum_{(i,j)\in K_1} |\gamma^h_{i,j}||u_{h,i}-u_{h,j}||v_i-v_j|\\
&\qquad +\eta\nu h^d \sum_{(i,j)\in K_1} |u_{h,i}-u_{h,j}||v_i-v_j|,\\
&\le \eta\left(\sum_{(i,j)\in K_1}|\gamma^h_{i,j}|(u_{h,j}-u_{h,i})^2\right)^{1/2}\left(\sum_{(i,j)\in K_1}|\gamma^h_{i,j}|(v_i-v_j)^2\right)^{1/2}\\
&\qquad +\eta\nu h^d\left(\sum_{(i,j)\in K_2}|u_{h,j}-u_{h,i}|^2\right)^{1/2}\left(\sum_{(i,j)\in K_2}|v_j-v_i|^2\right)^{1/2}\\
&\le C\eta\left(\sum_{(i,j)\in K}(-\gamma^h_{i,j})(u_{h,j}-u_{h,i})^2\right)^{1/2}\\
&\qquad \times\left(\sum_{i,j\in K}(-\gamma^h_{i,j})(v_i-v_j)^2\right)^{1/2}\\
&\qquad +2\eta\nu h^d\left(\sum_{(i,j)\in K_2}(u^2_{h,j}+u^2_{h,i})\right)^{1/2}\left(\sum_{(i,j)\in K_2}(v_j^2+v_i^2)\right)^{1/2}\\
&\le C\eta B(u_h,u_h)^{1/2}B(v,v)^{1/2}+4\eta\nu\kappa\|u_h\|_{L_2(\Omega)}\|v\|_{L_2(\Omega)}\\
&\le C\eta\|u\|_E\|v\|_E, \qquad (4.26)
\end{aligned}$$

for $v\in V_h$ with $\int_\Omega v\,dx = 0$. From the definition of $L(v)$ and $L^*(v)$,

$$\begin{aligned}
|L^*(v) - L(v)| &= |\sum_{i \in N_h} (f_i^{h*} + g_i^{h*}) v_i - \sum_{i \in N_h} (f_i^h + g_i^h) v_i| \\
&= |\sum_{i \in N_h} (f_i^{h*} - f_i^h) v_i + \sum_{i \in N'_h} (g_i^{h*} - g_i^h) v_i| \\
&\leq \left(\sum_{i \in N_h} |f_i^{h*} - f_i^h|^2 \right)^{1/2} \left(\sum_{i \in N_h} |v_i|^2 \right)^{1/2} \\
&+ \left(\sum_{i \in N'_h} |g_i^{h*} - g_i^h|^2 \right)^{1/2} \left(\sum_{i \in N'_h} |v_i|^2 \right)^{1/2} . \qquad (4.27)
\end{aligned}$$

Using estimates (4.15) and (4.4) we have

$$\begin{aligned}
\sum_{i \in N_h} |f_i^{h*} - f_i^h|^2 &= C\epsilon^2 \nu^2 \|f\|^2_{L_\infty(\Omega)} h^{2d} |N_h| \\
&= C\epsilon^2 \nu^2 \|f\|^2_{L_\infty(\Omega)} h^d \qquad (4.28)
\end{aligned}$$

and using (4.17) and (4.4) we have

$$\begin{aligned}
\sum_{i \in N'_h} |g_i^{h*} - g_i^h|^2 &\leq C\tau^2 \nu^2 \|g\|^2_{L_\infty(\Gamma)} h^{2d-2} |N'_h| \\
&\leq C\tau^2 \nu^2 \|g\|^2_{L_\infty(\Gamma)} h^{d-1}. \qquad (4.29)
\end{aligned}$$

From Axiom 3, the Poincaré inequality, and the trace inequality ([18]) we have

$$\sum_{i \in N_h} |v_i|^2 \leq Ch^{-d} \|v\|^2_{L_2(\Omega)} \leq Ch^{-d} \|v\|^2_E \qquad (4.30)$$

and

$$\sum_{i \in N'_h} |v_i|^2 \leq Ch^{1-d} \|v\|^2_{L_2(\Gamma)} \leq Ch^{1-d} \|v\|^2_{1,\Omega} \leq Ch^{1-d} \|v\|^2_E. \qquad (4.31)$$

Combining (4.28)–(4.31) we have

$$\sum_{i \in N_h} |f_i^{h*} - f_i^h|^2 \sum_{i \in N_h} |v_i|^2 \leq C\epsilon^2 \nu^2 \|f\|^2_{L_\infty(\Omega)} \|v\|^2_E$$

and

$$\sum_{i \in N'_h} |g_i^{h*} - g_i^h|^2 \sum_{i \in N'_h} |v_i|^2 \leq C\tau^2 \nu^2 \|g\|^2_{L_\infty(\Gamma)} \|v\|^2_E,$$

and thus, using (4.27), we obtain

$$|L^*(v) - L(v)| \leq C\nu[\epsilon \|f\|_{L_\infty(\Omega)} + \tau \|g\|_{L_\infty(\Gamma)}] \|v\|_E, \qquad (4.32)$$

Now combine (4.25), (4.26), and (4.32) to get

$$\begin{aligned}
|B^*(u_h^* - u_h, v)| &\leq |L^*(v) - L(v)| + |\frac{1}{2}Q(u_h, v)| \\
&\leq C\left[\epsilon\|f\|_{L_\infty(\Omega)} + \tau\|g\|_{L_\infty(\Gamma)} + \eta\|u\|_E\right]\|v\|_E,
\end{aligned}$$

for $v \in V_h$ with $\int_\Omega v\,dx = 0$. Since we can assume $v = u_h^* - u_h$ has zero average, we can let $v = u_h^* - u_h$ in this estimate and use Lemmas 4.3 and 4.4 to get

$$\begin{aligned}
\|u_h^* - u_h\|_E^2 &\leq CB^*(u_h^* - u_h, u_h^* - u_h) \\
&\leq C\left[\epsilon\|f\|_{L_\infty(\Omega)} + \tau\|g\|_{L_\infty(\Gamma)} + \eta\|u\|_E\right]\|u_h^* - u_h\|_E,
\end{aligned}$$

and hence

$$\|u_h^* - u_h\|_E \leq C\left[\epsilon\|f\|_{L_\infty(\Omega)} + \tau\|g\|_{L_2(\Gamma)} + \eta\|u\|_E\right], \ \forall h,$$

as desired. In using Lemma 4.4, we assume $|\eta| < \frac{2}{C_2 + 4\kappa\nu C_3^{-1}C_4}$, where η, ν, κ, C_2, C_3, and C_4 are as in Lemma 4.3.

□

Next we state and prove our first main error estimate.

Theorem 4.2 *Suppose our shape functions and our quadrature scheme satisfy Axioms 1–10. Then for small η, there is a constant C, independent of u, ϵ, τ, η, and h, such that*

$$\|u - u_h^*\|_E \leq C\left[h\|u\|_{2,\Omega} + \epsilon\|f\|_{L_\infty(\Omega)} + \tau\|g\|_{L_\infty(\Gamma)} + \eta\|u\|_E\right], \ \forall h. \qquad (4.33)$$

Proof. Write

$$u - u_h^* = (u - u_h) + (u_h - u_h^*).$$

Using Axiom (4.18) and Theorem 4.1 we have

$$\begin{aligned}
\|u - u_h^*\|_E &\leq \|u - u_h\|_E + \|u_h^* - u_h\|_E \\
&\leq C\left[h\|u\|_{2,\Omega} + \epsilon\|f\|_{L_\infty(\Omega)} + \tau\|g\|_{L_\infty(\Gamma)} + \eta\|u\|_E\right], \ \forall h.
\end{aligned}$$

□

The Correction Process

The second assumption in Axiom 8 — the Row Sum Condition, is unlikely to be satisfied (*cf.* the discussion at the end of Section 2). Likewise the second assumption in Axiom 9 — the Right-Hand Side Condition — is unlikely to hold, so we, in fact, do not have an estimates for $\|u - u_h^*\|_E$. Recall that for the example discussed in Section 3, $\|u - u_h^*\|_E$ behaves erratically, in particular is not small. As suggested in the last paragraph of Section 3, we consider a corrected stiffness matrix γ^{h**} and corrected right-hand side vectors f^{h**} and g^{h**}, which do satisfy Axioms 8 and 9, and then compute the corresponding approximate solution, u_h^{**}, from system (2.14), with $\gamma_{i,j}^{h*}, f_i^{h*}, g_i^{h*}$ replaced by $\gamma_{i,j}^{h**}, f_i^{h**}, g_i^{h**}$, respectively. We then apply Theorems 4.1 and 4.2 to the corrected problem to estimate $\|u_h - u_h^{**}\|_E$ and $\|u - u_h^{**}\|_E$. To do this we need to show that $\gamma^{h**}, f^{h**}, g^{h**}$ satisfy the Axioms involving γ^{h*}, f^{h*}, g^{h*}, namely Axioms 7–10.

We consider the following specific corrections: We define γ^{h**} by letting

$$\gamma_{i,j}^{h**} = \gamma_{i,j}^{h*}, \text{ for } i \neq j \tag{4.34}$$

and

$$\gamma_{i,i}^{h**} = -\sum_{j \in N_h, j \neq i} \gamma_{i,j}^{h*}. \tag{4.35}$$

If $\|g\|_{L_\infty(\Gamma)} \leq \|f\|_{L_\infty(\Omega)}$, we define

$$f_i^{h**} = f_i^{h*} - \frac{\sum_{j \in N_h} f_j^{h*} + \sum_{j \in N_h'} g_j^{h*}}{|N_h|} \text{ and } g_i^{h**} = g_i^{h*}; \tag{4.36}$$

we correct f_i^{h*}, but not g_i^{h*}. On the other hand, if $\|f\|_{L_\infty(\Omega)} < \|g\|_{\infty(\Gamma)}$, we define

$$g_i^{h**} = g_i^{h*} - \frac{\sum_{j \in N_h} f_j^{h*} + \sum_{j \in N_h'} g_j^{h*}}{|N_h'|} \text{ and } f_i^{h**} = f_i^{h*}; \tag{4.37}$$

in this case we correct g_i^{h*}, but not f_i^{h*}.

It is immediate that γ^{h**} is symmetric, *i.e.*, Axiom 7 is satisfied. It follows directly from the definition of γ^{**} in (4.34) and (4.35) that $\sum_{i \in N_h} \gamma_{i,j}^{h**} = 0$, *i.e.*, γ^{h**} satisfies Axiom 8. If $\|g\|_{L_\infty(\Gamma)} \leq \|f\|_{L_\infty(\Omega)}$, then

$$\sum_{i \in N_h} f_i^{h**} + \sum_{i \in N_h'} g_i^{h**} = \sum_{i \in N_h} f_i^{h*} - \sum_{i \in N_h} f_i^{h*} - \sum_{i \in N_h'} g_i^{h*} + \sum_{i \in N_h'} g_i^{h*} = 0;$$

likewise $\sum_{i \in N_h} f_i^{h**} + \sum_{i \in N_h'} g_i^{h**} = 0$, when $\|f\|_{L_\infty(\Omega)} < \|g\|_{\infty(\Gamma)}$. So the right-hand side vectors f^{h**}, g^{h**} satisfy the Axiom 9.

It remains to consider Axiom 10. First consider γ^{h**}. Since $\gamma_{i,j}^{h**} = \gamma_{i,j}^{h}$ for $i \neq j$, it is immediate that Axiom 10 holds for $\gamma_{i,j}^{h**}$ with the same η when $i \neq j$. For $i = j$, using the definition of γ^{h**} in (4.34) and (4.35) and Axioms 8 and 10, we have

$$\begin{aligned}
\gamma_{i,i}^{h**} - \gamma_{i,i}^{h} &= -\sum_{j \in N_h, j \neq i} \gamma_{i,j}^{h*} - \gamma_{i,i}^{h} \\
&= -\sum_{j \in N_h, j \neq i} (\gamma_{i,j}^{h} + \eta_{i,j}^{h}) - \gamma_{i,i}^{h} \\
&= -\sum_{i \in N_h} \gamma_{i,j}^{h} - \sum_{j \in N_h, j \neq i} \eta_{i,j}^{h} \\
&= -\sum_{j \in N_h, j \neq i} \eta_{i,j}^{h}.
\end{aligned}$$

Let

$$S_i^1 = \{j : (i,j) \in K_1\} \text{ and } S_i^2 = \{j : (i,j) \in K_2\}.$$

Then, using Axioms 4 and 6 and estimate (4.13), we have

$$\begin{aligned}
\left|\gamma_{i,i}^{h**} - \gamma_{i,i}^{h}\right| &\le \sum_{j\in S_i^1} |\eta_{i,j}^h| + \sum_{j\in S_i^2} |\eta_{i,j}^h| \\
&\le \eta \sum_{j\in S_i^1} |\gamma_{i,j}^h| + \eta\nu h^d \sum_{j\in S_i^2} 1 \\
&\le \eta C_5 |\gamma_{i,i}^h||S_i^1| + \eta\nu h^d |S_i^2| \\
&\le \eta C_5 |\gamma_{i,i}^h|\kappa + \eta\nu h^d \kappa \\
&\le \max(C_5,1)\eta\kappa \max(|\gamma_{i,i}^h|, \nu h^d),
\end{aligned}$$

where C_5 is the constant in Axiom 6. Thus $\gamma_{i,j}^{h**}$ satisfies Axiom 10 for $i=j$, with η replaced by $\max(C_5,1)\kappa\eta$.

Next consider the right-hand side vectors. Suppose $\|g\|_{L_\infty(\Gamma)} \le \|f\|_{L_\infty(\Omega)}$. Then, since $g_i^{h**} = g_i^{h*}$, it is immediate from Axiom 10 that $|g_i^{h**} - g_i^h| = |g_i^{h*} - g_i^h| \le \tau \max(|g_i^h|, \nu h^{d-1}\|g\|_{L_2(\Gamma)})$, so g_i^{h**} satisfies Axiom 10 with the same τ. We next estimate $|f_i^{h**} - f_i^h|$. Using the definition of f_i^{h**} in (4.36), Axioms 5, 10, 5, and estimates (4.17), and (4.14), we obtain

$$\begin{aligned}
|f_i^{h**} - f_i^h| &= \left|(f_i^{h*} - f_i^h) - \frac{\sum_{j\in N_h} f_j^{h*} + \sum_{j\in N_h'} g_j^{h*}}{|N_h|}\right| \\
&= \left|\epsilon_i^h - \frac{\sum_{j\in N_h}(f_j^h + \epsilon_j^h) + \sum_{j\in N_h'}(g_j^h + \tau_j^h)}{|N_h|}\right| \\
&= \left|\epsilon_i^h - \frac{\sum_{j\in N_h}\epsilon_j^h}{|N_h|} - \frac{\sum_{j\in N_h'}\tau_j^h}{|N_h|}\right| \\
&\le |\epsilon_i^h| + \frac{\sum_{j\in N_h}|\epsilon_j^h|}{|N_h|} + \frac{\sum_{j\in N_h'}|\tau_j^h|}{|N_h|} \\
&\le \epsilon \max(|f_i^h|, \nu h^d \|f\|_{L_\infty(\Omega)}) \\
&\qquad + \epsilon \frac{\sum_{j\in N_h} \max(|f_j^h|, \nu h^d \|f\|_{L_\infty(\Omega)})}{|N_h|} \\
&\qquad + \tau \frac{\sum_{j\in N_h'} \max(|g_j^h|, \nu h^{d-1} \|g\|_{L_\infty(\Gamma)})}{|N_h|} \\
&\le \epsilon \max(|f_i^h|, \nu h^d \|f\|_{L_\infty(\Omega)}) \\
&\qquad + \epsilon \frac{\max(|f_j^h| : j \in N_h, \nu h^d \|f\|_{L_\infty(\Omega)})|N_h|}{|N_h|} \\
&\qquad + \tau \frac{\max(|g_j^h| : j \in N_h', \nu h^{d-1} \|g\|_{L_\infty(\Gamma)})|N_h'|}{|N_h|},
\end{aligned} \tag{4.38}$$

and hence

$$
\begin{aligned}
|f_i^{h**} - f_i^h| \quad \leq \quad & \epsilon \max(|f_i^h|, \nu h^d \|f\|_{L_\infty(\Omega)}) \\
& \qquad + \epsilon \max(C_1 \nu h^d \|f\|_{L_\infty(\Omega)}, \nu h^d \|f\|_{L_\infty(\Omega)}) \\
& \qquad\qquad + \tau \max(C_1 \nu h^{d-1} \|g\|_{L_\infty(\Gamma)}, \nu h^{d-1} \|g\|_{L_\infty(\Gamma)}) rh \\
\leq \quad & \epsilon \max(|f_i^h|, \nu h^d \|f\|_{L_\infty(\Omega)}) + \epsilon \nu h^d \|f\|_{L_\infty(\Omega)} \max(1, C_1) \\
& \qquad\qquad + \tau \, r \nu h^d \|g\|_{L_\infty(\Gamma)} \max(1, C_1) \\
\leq \quad & [\epsilon + (\epsilon + \tau) \max(1, r, C_1, rC_1)] \max(|f_i^h|, \nu h^d \|f\|_{L_\infty(\Omega)}),
\end{aligned}
$$

where C_1 is the constant introduced in Remark 4.10 and r is the constant in estimate (4.7). This shows that f_i^{h**} satisfies Axiom 10 with ϵ replaced by $\epsilon + (\epsilon + \tau) \max(1, r, C_1, rC_1)$.

Next we suppose $\|f\|_{L_\infty(\Omega)} < \|g\|_{L_\infty(\Gamma)}$. In this case it is immediate that f_i^{h**} satisfies Axiom 10 with the same ϵ, and we find that g_i^{h**} satisfies Axiom 10 with τ replaced by $\tau + (\epsilon + \tau) \max(1, \tilde{r}, C_1, \tilde{r}C_1)$, where $\tilde{r}$ is the constant introduced in estimate (4.7).

Error Estimates for the Corrected Approximation

Error estimates can be obtained by applying Theorems 4.1 and 4.2 to the corrected problem. In effect, we are considering Section 4 with these replacements. In particular, we obtain Theorems 4.1 and 4.2 with these replacements, and, following the above discussion, obtain

Theorem 4.3 *Suppose our shape functions and our quadrature scheme satisfy Axioms 1–10. Then for small η, there is a constant C, independent of u, ϵ, τ, η, and h, such that*

$$\|u_h - u_h^{**}\|_E \leq C \left[\eta \|u\|_E + (\epsilon + \tau) \|f\|_{L_\infty(\Omega)} + (\epsilon + \tau) \|g\|_{L_\infty(\Gamma)}\right], \ \forall h. \quad (4.39)$$

Theorem 4.4 *Suppose our shape functions and our quadrature scheme satisfy Axioms 1–10. Then for small η, there is a constant C, independent of u, ϵ, τ, η, and h, such that*

$$\|u - u_h^{**}\|_E \leq C \left[h \|u\|_{2,\Omega} + \eta \|u\|_E + (\epsilon + \tau) \|f\|_{L_\infty(\Omega)} + (\tau + \epsilon) \|g\|_{L_\infty(\Gamma)}\right], \ \forall h. \quad (4.40)$$

In Figure 4.1 we present log-log plots of the relative errors $\frac{\|u_h - u_h^{**}\|_E}{\|u\|_E}$ and $\frac{\|u - u_h^{**}\|_E}{\|u\|_E}$ with respect to h for the one dimensional problem discussed in Section 3. The stiffness matrix γ_h^* is computed with the same quadrature methods as in Section 3, but the matrix is then corrected to satisfy the Row Sum Condition (4.10), and denoted by γ_h^{**}. The right-hand side vector $f_i^h + g_i^h$ has been computed exactly ($\epsilon = \tau = 0$).

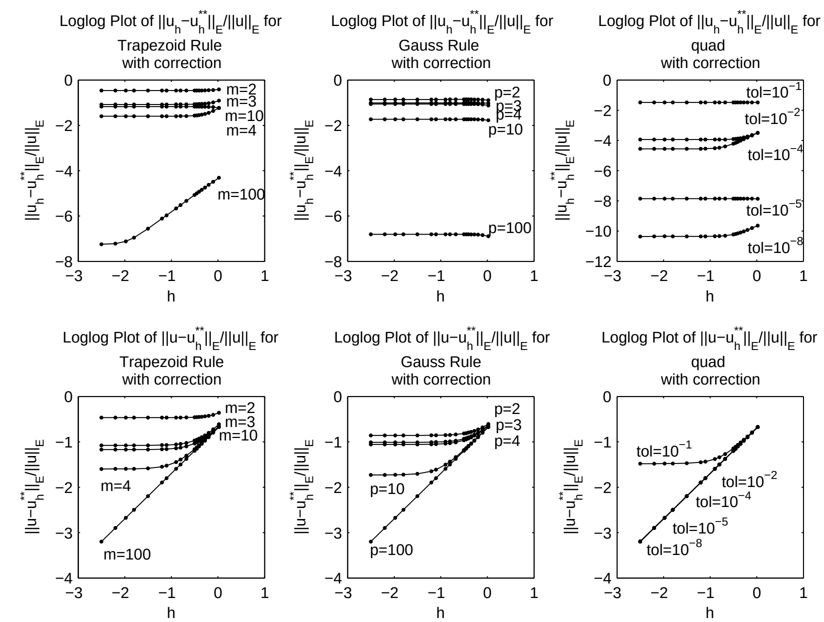

Figure 4.1: The log-log plot of $\frac{\|u_h - u_h^{**}\|_E}{\|u\|_E}$ and $\frac{\|u - u_h^{**}\|_E}{\|u\|_E}$ with respect to h with correction for various quadrature schemes. For quadrature schemes, we used the m-panel Trapezoid Rule, the p-point Gauss Rule, and MATLAB's `quad` (adaptive Simpson quadrature) with different tolerances, `tol`. The relative error $\frac{\|u_h - u_h^{**}\|_E}{\|u\|_E}$ becomes nearly constant as $h \to 0$, and this constant reflects the accuracy of the quadrature (η). The relative error $\frac{\|u - u_h^{**}\|_E}{\|u\|_E}$ first decreases with decreasing h, and then approaches a constant as $h \to 0$. This figure should be compared with Fig. 3.1.

We observe that the relative error $\|u_h - u_h^{**}\|_E / \|u\|_E$ becomes nearly constant as $h \to 0$; this constant reflects the accuracy of the quadrature (η). On the other hand, the relative error $\|u - u_h^{**}\|_E / \|u\|_E$ first decreases with decreasing h and then levels off, becoming nearly constant as $h \to 0$. These figures and estimate (4.40) indicate that the error has two components: one due to the MM approximation (see the estimate of $\|u - u_h\|_E$ in the proofs of Theorems 4.2 and Theorem 4.4) and the other due to quadrature (see estimate (4.39)). From (4.40) and the second row in Fig. 4.1 we see that for given quadrature accuracy η, ϵ, and τ, the error for small h is completely governed by the quadrature accuracy. We have to set η, ϵ, and τ equal to $o(1)$ if we want the relative error to converge, and we have to set η, ϵ, and τ equal to $O(h)$ if we want the relative error to be

$O(h)$.

So far we have discussed the one dimensional problem (3.1). Now consider the two dimensional problem

$$\left\{ \begin{array}{l} -\Delta u = \cos x \cos y, (x,y) \in \Omega = [0,\pi] \times [0,\pi] \\ \frac{\partial u}{\partial n}(x,y) = 0 \text{ for } (x,y) \in \partial\Omega \end{array} \right. . \tag{4.41}$$

For an MM basis we take tensor products of the ϕ_j^h used in the one dimensional problem.

In Fig 4.2 we present log-log plots of the relative errors $\|u - u_h^*\|_E/\|u\|_E$ for the p-point Gauss Rule without correction and $\|u - u_h^{**}\|_E/\|u\|_E$ and $\|u_h - u_h^{**}\|_E/\|u\|_E$ for p-point Gauss Rule with correction. We have computed the right-hand side exactly ($\eta = \tau = 0$).

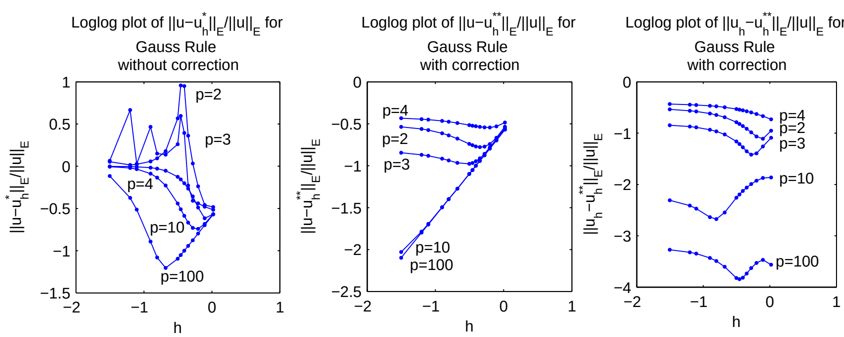

Figure 4.2: The log-log plot of $\frac{\|u-u_h^*\|_E}{\|u\|_E}$ for p-point Gauss Rule without correction and $\frac{\|u-u_h^{**}\|_E}{\|u\|_E}$ and $\frac{\|u_h-u_h^{**}\|_E}{\|u\|_E}$ for p-point Gauss Rule with correction for the two dimensional problem (4.41). This figure should be compared with Fig. 4.1.

The error behavior is similar to that of the one dimensional problem. It is dimension independent, as the theory predicted.

5 The General Case

In this section we consider the problem (2.1) with $\alpha = 1$, *i.e.*, we consider

$$\left\{ \begin{array}{l} -\Delta u + u = f \text{ in } \Omega \\ \frac{\partial u}{\partial n} = g \text{ on } \Gamma = \partial\Omega \end{array} \right. , \tag{5.1}$$

with variational formulation:

Find $u \in H^1(\Omega)$ satisfying

$$B(u,v) \equiv B_1(u,v) + B_0(u,v) = L(v), \text{ for all } v \in H^1(\Omega), \tag{5.2}$$

where

$$B_1(u,v) \equiv \int_\Omega \nabla u \cdot \nabla v dx, \quad B_0(u,v) = \int_\Omega uv\, dx \tag{5.3}$$

and

$$L(v) \equiv \int_\Omega fv\, dx + \int_\Gamma gv\, ds.$$

B_1 is the same as B and L is the same as L in Section 2. The solution exists and is unique (the hypothesis $L(1) = 0$ is not needed here). The Energy Space associated with problem (5.1) is $H_E = H^1(\Omega)$ and $\|u\|_E = \|u\|_{1,\Omega}$ is the Energy Norm.

Define

$$m_{i,j}^h = B_0(\phi_i^h, \phi_j^h) = \int_{\omega_i^h \cap \omega_j^h} \phi_i^h \phi_j^h\, dx,$$

where $\{\phi_i^h\}_{i=1}^{N_h}$ are shape functions for our meshless method. Then, from Axiom 5 we have

$$|m_{i,j}^h| \le \nu^2 |\omega_i^h \cap \omega_j^h| \le Ch^d, \forall i, j. \tag{5.4}$$

By Axiom 4, for $i \in N_h$, $m_i \le \kappa$ of the indices j are such that $\omega_i^h \cap \omega_j^h \ne \emptyset$; denote these by $j_1^i, j_2^i, \cdots, j_{m_i}^i$. Then

$$\omega_i^h \cap \omega_j^h = \emptyset \text{ and hence } m_{i,j}^h = 0, \text{ for } j \ne j_1^i, j_2^i, \cdots, j_{m_i}^i. \tag{5.5}$$

From the definition of $B_0(w,v)$ we see that

$$B_0(w,v) = \sum_{i,j \in N_h} m_{i,j}^h w_i v_j, \forall w = \sum_{i \in N_h} w_i \phi_i^h \in V_h, v = \sum_{i \in N_h} v_i \phi_i^h \in V_h. \tag{5.6}$$

We then naturally define

$$B_0^*(w,v) \equiv \sum_{i,j \in N_h} m_{i,j}^{h*} w_i v_j, \tag{5.7}$$

where

$$m_{i,j}^{h*} = \int\!\!\!\!\!\!-_{\omega_i^h \cap \omega_j^h} \phi_i^h \phi_j^h\, dx.$$

Regarding $m_{i,j}^{h*}$, we assume

- **Axiom 11**

$$m_{i,j}^{h*} = m_{i,j}^h + \delta_{i,j}^h, \tag{5.8}$$

with

$$|\delta_{i,j}^h| \le \delta \max(|m_{i,j}^h|, \nu^2 h^d).$$

Remark 5.1 δ in Axiom 11 is based on typical accuracy estimates when adaptive quadrature with tolerance δ is used.

Here are some consequences of Axiom 11:

Recall that

$$K = \{(i,j) : i,j \in N_h\} \text{ and } K_0 = \{(i,j) : \omega_i^h \cap \omega_j^h = \emptyset\}.$$

Then let

$$G_1 = \{(i,j) : (i,j) \notin K_0, |m_{i,j}| \geq \nu^2 h^d\}$$

and

$$G_2 = \{(i,j) : (i,j) \notin K_0, |m_{i,j}| < \nu^2 h^d\}.$$

With these definitions, K, G_0, and G_2 are pairwise disjoint, $K = K_0 \cup G_1 \cup G_2$,

$$m_{i,j}^h = \delta_{i,j}^h = 0,\ \forall (i,j) \in K_0,$$

and

$$|\delta_{i,j}^h| \leq \delta |m_{i,j}^h|,\ \forall (i,j) \in G_1 \text{ and } |\delta_{i,j}^h| \leq \delta \nu^2 h^d,\ \forall (i,j) \in G_2.$$

Define

$$B^*(w,v) = B_1^*(w,v) + B_0^*(w,v), \tag{5.9}$$

where, as in Section 2,

$$B_1^*(u,v) = \sum_{i,j \in N_h} \gamma_{i,j}^{h*} u_i v_j \text{ and } L^*(v) = \sum_{i \in N_h} f_i^{h*} v_i + \sum_{i \in N_h'} g_i^{h*} v_i, \tag{5.10}$$

with $\gamma_{i,j}^{h*}, f_i^{h*}$, and g_i^{h*} defined in (2.10) and (2.11).

Next we prove several lemmas.

Lemma 5.1 *There is a constant C, independent of w, v, and h, such that*

$$|B_0(w,v)| \leq \sum_{i,j \in N_h} |m_{i,j}^h||w_i||v_j| \leq C\|w\|_{L_2(\Omega)}\|v\|_{L_2(\Omega)}, \tag{5.11}$$

for $w = \sum_{i \in N_h} w_i \phi_i^h$ and $v = \sum_{i \in N_h} v_i \phi_i^h \in V_h$.

Remark 5.2 We note that $|B_0(w,v)| \leq \|w\|_{L_2(\Omega)}\|v\|_{L_2(\Omega)}$ (*cf.* the 1st and 3rd member of the inequality (5.11) with $C = 1$) follows directly from the definition of B_0. Also, the first inequality in (5.11) follows immediately from (5.6).

Proof. It remains to prove the second inequality in estimate (5.11). Using Axioms 3 and 4, (5.4), and (5.5), we have

$$
\begin{aligned}
\sum_{i,j\in N_h} |m^h_{i,j}||w_i||v_j| &\leq \sum_{i\in N_h} |w_i| \sum_{j=j^i_1}^{j^i_{m_i}} |m^h_{i,j}||v_j| \\
&= C \sum_{i\in N_h} h^{d/2}|w_i| \sum_{j=j^i_1}^{j^i_{m_i}} h^{d/2}|v_j| \\
&\leq C\kappa^{1/2} \sum_{i\in N_h} h^{d/2}|w_i| \left[\sum_{j=j^i_1}^{j^i_{m_i}} h^d |v_j|^2 \right]^{1/2} \\
&\leq C\kappa^{1/2} \left[\sum_{i\in N_h} h^d |w_i|^2 \right]^{1/2} \left[\sum_{i\in N_h} \sum_{j=j^i_1}^{j^i_{m_i}} h^d |v_j|^2 \right]^{1/2} \\
&\leq C\kappa \left[\sum_{i\in N_h} h^d |w_i|^2 \right]^{1/2} \left[\sum_{i\in N_h} h^d |v_i|^2 \right]^{1/2} \\
&\leq C\|w\|_{L_2(\Omega)}\|v\|_{L_2(\Omega)}.
\end{aligned}
$$

□

Lemma 5.2 *There is a constant C, independent of δ, w, v, and h, such that*

$$
|B^*_0(w,v)| \leq [1 + C\delta]\, \|w\|_{L_2(\Omega)}\|v\|_{L_2(\Omega)}, \tag{5.12}
$$

for $w = \sum_{i=1}^{N_h} w_i\phi^h_i$ and $v = \sum_{i=1}^{N_h} v_i\phi^h_i \in V_h$.

Proof. Using (5.6), (5.7), and Axiom 11 we have

$$
\begin{aligned}
B^*_0(w,v) &= \sum_{i,j\in N_h} m^{h*}_{i,j} w_i v_j \\
&= \sum_{i,j\in N_h} m^h_{i,j} w_i v_j + \sum_{i,j\in N_h} \delta_{i,j} w_i v_j \\
&= B_0(w,v) + Q_0(w,v),
\end{aligned} \tag{5.13}
$$

where

$$
Q_0(w,v) = \sum_{i,j\in N_h} \delta_{i,j} w_i v_j.
$$

Using Axiom 11 and Lemma 5.1 and its proof, we have

$$\begin{aligned}|Q_0(w,v)| &= |\sum_{(i,j)\in G_1}\delta^h_{i,j}w_iv_j+\sum_{(i,j)\in G_2}\delta^h_{i,j}w_iv_j| \\ &\le \delta\sum_{(i,j)\in K}|m^h_{i,j}||w_i||v_j|+\delta\nu^2h^d\sum_{(i,j)\in K}|w_i||v_j| \\ &\le \delta C\|w\|_{L_2(\Omega)}\|v\|_{L_2(\Omega)}+C\delta\nu^2\|w\|_{L_2(\Omega)}\|v\|_{L_2(\Omega)} \\ &\le C\delta\|w\|_{L_2(\Omega)}\|v\|_{L_2(\Omega)}, \qquad (5.14)\end{aligned}$$

and thus, using (5.13) and Lemma 5.1,

$$|B^*_0(w,v)|\le(1+\delta C)\|w\|_{L_2(\Omega)}\|v\|_{L_2(\Omega)}.$$

□

Lemma 5.3 *For δ sufficiently small, there is a positive constant C, independent of δ, w and h, such that*

$$B^*_0(w,w)\ge[1-C\delta]\|w\|^2_{L_2(\Omega)}, \qquad (5.15)$$

for $w=\sum_{i\in N_h}w_i\phi^h_i\in V_h$.

Proof. Using (5.13) and (5.14) we have

$$\begin{aligned}B^*_0(w,w) &= B_0(w,w)+\sum_{i,j\in N_h}\delta_{i,j}w_iw_j \\ &\ge \|w\|^2_{L_2(\Omega)}-|Q_0(w,v)| \\ &\ge (1-C\delta)\|w\|^2_{L_2(\Omega)},\end{aligned}$$

for δ sufficiently small.

□

Next we prove our error estimates. The MM approximation u_h to the solution u of (5.1) is characterized by

$$\begin{cases} u_h\in V_h \\ B(u_h,v)=L(v),\ \forall v\in V_h,\end{cases} \qquad (5.16)$$

and the quadrature approximation u^*_h by

$$\begin{cases} u^*_h\in V_h \\ B^*(u^*_h,v)=L^*(v),\ \forall v\in V_h,\end{cases} \qquad (5.17)$$

where $B, B_1, B_0, B^*, B^*_1, B^*_0$, and L^* are defined in (5.2), (5.3), (5.7), (5.9), and (5.10).

Before stating and proving our error estimates, it will be helpful to summarize our assumptions: We assume Axioms 2 and 6 for γ^h, the exact stiffness matrix associated with the form B_1. We assume Axioms 1, 3, 4, and 5, which are assumptions on the basis $\{\phi_i^h : i \in N_h\}$. We assume Axioms 7 and 8, symmetry and the Row Sum Condition for γ^{h*}, the quadrature matrix associated with the form B_1. We do not assume the 2nd equation in Axiom 9. Finally, we assume Axiom 10, the quadrature assumptions on $\int_{\omega_i^h \cap \omega_j^h} \nabla\phi_i^h \cdot \nabla\phi_j^h \, dx$, $\int_{\omega_i^h} f\phi_i^h \, dx$, and $\int_{\Gamma\cap\omega_i^h} g\phi_i^h \, ds$, and Axiom 11, the quadrature assumption on $\int_{\omega_i^h \cap \omega_j^h} \phi_i^h \phi_j^h \, dx$.

Lemma 4.4, applied to B_1^*, shows that

$$B_1^*(w,w) \geq C|w|_{1,\Omega}^2, \forall w \in V_h, \text{ for } \eta \text{ small.}$$

Combining this with Lemma 5.3, we obtain

$$B^*(w,w) \geq C(\eta,\delta)\|w\|_{1,\Omega}^2 = C(\eta,\delta)\|w\|_E^2, \forall w \in V_h, \tag{5.18}$$

with $C(\eta,\delta) > 0$, for η and δ small. It follows from (5.18) that u_h^*, the solution of (5.17), exists and is unique. As pointed out above, we do not need to assume the compatibility condition $\sum_{i\in N_h} f_i^{h*} + \sum_{i\in N_h'} g_i^{h*} = 0$ here. Using Lemma 4.3 applied to B_1^*, and Lemma 5.2 we see that

$$B^*(w,v) \leq C\|w\|_E\|v\|_E, \forall w \in V_h. \tag{5.19}$$

Remark 5.3 Estimates (5.18) and (5.19) show the coercivity and boundedness of B^*.

Theorem 5.1 *There is a constant C, independent of u, η, ϵ, τ, and h, such that*

$$\|u_h - u_h^*\|_E \leq C\left[\epsilon\|f\|_{L_\infty(\Omega)} + \tau\|g\|_{L_\infty(\Gamma)} + (\eta+\delta)\|u\|_E\right], \tag{5.20}$$

for η, δ small.

Proof. As in the proof of Theorem 4.1, we have

$$B_1^*(u_h,v) = B_1(u_h,v) - \frac{1}{2}Q_1(u_h,v), \tag{5.21}$$

where

$$Q_1(u_h,v) = \sum_{i,j\in N_h} \eta_{i,j}^h(u_{h,i} - u_{h,j})(v_i - v_j),$$

and

$$|Q_1(u_h,v)| \leq C\eta|u_h|_{1,\Omega}|v|_{1,\Omega}. \tag{5.22}$$

Similarly, as in the proof of Lemma 5.2, specifically from (5.13) and (5.14),

$$B_0^*(u_h,v) = B_0(u_h,v) + Q_0(u_h,v), \tag{5.23}$$

where

$$|Q_0(u_h,v)| \leq C\delta\|u_h\|_{L_2(\Omega)}\|v\|_{L_2(\Omega)}. \tag{5.24}$$

From (5.2), (5.9), (5.13), (5.16), (5.17), (5.21), and (5.23) we obtain

$$\begin{aligned} B^*(u_h^* - u_h, v) &= B^*(u_h^*, v) - B^*(u_h, v) \\ &= L^*(v) - B_1^*(u_h, v) - B_0^*(u_h, v) \\ &= L^*(v) - [B_1(u_h, v) - \frac{1}{2}Q_1(u_h, v)] - [B_0(u_h, v) + Q_0(u_h, v)] \\ &= L^*(v) - B(u_h, v) + \frac{1}{2}Q_1(u_h, v) - Q_0(u_h, v) \\ &= L^*(v) - L(v) + \frac{1}{2}Q_1(u_h, v) - Q_0(u_h, v). \end{aligned}$$

Thus, using (4.32), (5.22), (5.24), and the energy estimate, $\|u_h\|_E \leq \|u\|_E$, we obtain

$$\begin{aligned} |B^*(u_h^* - u_h, v)| &\leq |L^*(v) - L(v)| + \frac{1}{2}|Q_1(u_h, v)| + |Q_0(u_h, v)| \\ &\leq C[\epsilon\|f\|_{L_\infty(\Omega)}\|v\|_E + \tau\|g\|_{L_\infty(\Gamma)}\|v\|_E \\ &\qquad + \eta|u_h|_{1,\Omega}|v|_{1,\Omega} + \delta\|u_h\|_{L_2(\Omega)}\|v\|_{L_2(\Omega)}] \\ &\leq C\left[\epsilon\|f\|_{L_\infty(\Omega)} + \tau\|g\|_{L_\infty(\Gamma)} + (\eta+\delta)\|u\|_E\right]\|v\|_E. \end{aligned}$$

Combining this estimate and with (5.18) and (5.19) we have

$$\begin{aligned} \|u_h^* - u_h\|_E^2 &\leq CB^*(u_h^* - u_h, u_h^* - u_h) \\ &\leq C\left[\epsilon\|f\|_{L_\infty(\Omega)} + \tau\|g\|_{L_\infty(\Gamma)} + (\eta+\delta)\|u\|_E\right]\|u_h^* - u_h\|_E, \end{aligned}$$

and hence

$$\|u_h^* - u_h\|_E \leq C\left[\epsilon\|f\|_{L_\infty(\Omega)} + \tau\|g\|_{L_\infty(\Gamma)} + (\eta+\delta)\|u\|_E\right],$$

for small η, δ, as desired □

Theorem 5.2 *There is a constant C, independent of $u, \eta, \epsilon, \tau, \delta$, and h, such that*

$$\|u - u_h^*\|_E \leq C\left[h\|u\|_{2,\Omega} + \epsilon\|f\|_{L_\infty(\Omega)} + \tau\|g\|_{L_\infty(\Gamma)} + (\eta+\delta)\|u\|_E\right], \ \forall h. \quad (5.25)$$

Proof. This result follows for Theorem 5.1 as Theorem 4.2 follows from Theorem 4.1. □

The Correction Process

For the general boundary value problem (5.1) we will need to consider the correction process. Since we assume the Row Sum Condition only for the part of the stiffness matrix corresponding to the form B_1, we need only correct this part; specifically, we correct this part of the stiffness matrix using (4.34) and (4.35). Since we do not need to assume the Right-Hand Side Sum Condition, we do not correct the right-hand side vectors. We again denote the approximate solution of the corrected problem by u_h^{**}. Here are the estimates for $\|u_h - u_h^{**}\|_E$ and $\|u - u_h^{**}\|_E$.

Theorem 5.3 *There is a constant C, independent of $u, \eta, \epsilon, \tau, \delta$, and h, such that*

$$\|u_h - u_h^{**}\|_E \leq C\left[(\epsilon + \tau)\|f\|_{L_\infty(\Omega)} + (\tau + \epsilon)\|g\|_{L_\infty(\Gamma)} + (\eta + \delta)\|u\|_E\right], \; \forall h. \tag{5.26}$$

Theorem 5.4 *There is a constant C, independent of $u, \eta, \epsilon, \tau, \delta$, and h, such that*

$$\begin{aligned}\|u - u_h^{**}\|_E \quad \leq \quad & C[h\|u\|_{2,\Omega} + (\epsilon + \tau)\|f\|_{L_\infty(\Omega)} + (\tau + \epsilon)\|g\|_{L_\infty(\Gamma)} \\ & + (\eta + \delta)\|u\|_E], \; \forall h.\end{aligned} \tag{5.27}$$

6 Remarks and Conclusions

We have developed a theoretical framework to analyze the effect of numerical integration on meshless methods. We summarize below our main results:

- We have identified a simple correction procedure in the computed stiffness matrix to avoid the major problems due to the use of numerical integration. Under the assumption that this correction has been made, we have proved error estimates.
- Our result indicate that we need to require increased quadrature accuracy (*i.e.*, smaller values of the parameters η, ϵ, and τ) as $h \to 0$. This is in contrast to the situation with standard FEM, where the same quadrature scheme can be used on each element as $h \to 0$.

These results were illuminated with computational examples.

We have considered scalar boundary value problems with constant coefficients in this paper, but our results can be extended to general coercive, Neumann problems with non-constant coefficients.

Quadrature schemes for the Dirichlet problem and for MM of higher order, *i.e,* MM based on shape functions that reproduce polynomials of order higher that 1, will be addressed in a future paper.

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