ARTICLE IN PRESS

Comput. Methods Appl. Mech. Engrg. xxx (2009) xxx–xxx

Contents lists available at ScienceDirect

Comput. Methods Appl. Mech. Engrg.

journal homepage: www.elsevier.com/locate/cma

Effect of numerical integration on meshless methods

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ARTICLE INFO

Article history:
Received 1 December 2008
Accepted 17 April 2009
Available online xxxx

Keywords:
Galerkin methods
Meshless methods
Quadrature
Numerical integration
Error estimates

ABSTRACT

In this paper, we present the effect of numerical integration on meshless methods with shape functions that reproduce polynomials of degree $k \geqslant 1$. The meshless method was used on a second order Neumann problem and we derived an estimate for the energy norm of the error between the exact solution and the approximate solution from the meshless method under the presence of numerical integration. This estimate was obtained under the assumption that the numerical integration scheme satisfied a form of Green's formula. We also indicated how to obtain numerical integration schemes satisfying this property.

1. Introduction

Meshless methods (MM) were developed in early 1990s for numerically solving partial differential equations (PDE). This initiative was stimulated by the difficulties in mesh generation when available methods, e.g., the Finite Element Method (FEM), were used to solve various complex problems in engineering.

It was recognized from the very beginning of the development of MM that numerical integration posed bigger challenge in this method than the FEM, and the issue was discussed in various engineering papers, e.g., [5,8,9,11,12,10,14–16]. In FEM, the shape functions are piecewise polynomials of degree $k \geqslant 1$ and a careful mathematical analysis of the effect of numerical integration in FEM was published 30 years ago in [13]. The analysis required that the numerical integration in FEM, when applied to PDEs with constant coefficients, must evaluate the stiffness matrix exactly. This is easily achieved since the integrands of the elements of the stiffness matrix of FEM are polynomials of degree $2k-2$. The analysis also exploited the fact that ℓth order derivatives of the shape functions vanish locally (on each triangle) for $\ell \geqslant k+1$. In contrast, the shape functions used in MM are not piecewise polynomials and their ℓth order derivatives grow with ℓ. Moreover, the stiffness matrix cannot be evaluated exactly for PDEs with constant coefficients. Thus the shape functions used in MM lack the two most important features of the shape functions of FEM. Numerical integration in MM is a bigger challenge primarily because of the lack of these features.

Many interesting ideas on the use of numerical integration in MM were presented in the engineering papers mentioned above, but to the best of our knowledge, a careful mathematical analysis of the effect of numerical integration in MM was first reported in [4]. It is shown in this paper that the error in the approximate solution, obtained from MM with standard numerical quadrature, does not converge. It is then shown that if the stiffness matrix satisfies a condition referred to as the *zero row sum condition*, the energy norm of the error in the approximate solution is $O(h+\bar{\eta})$, where h is the standard discretization parameter related to the diameters of the supports of the shape functions and $\bar{\eta}$ is the parameter indicating the accuracy of the underlying numerical quadrature. Thus MM, with numerical integration, does not yield optimal order of convergence unless $\bar{\eta} = O(h)$. However, the analysis in [4] uses an assumption on the approximation space that is difficult to verify. We further note that the analysis is restricted to MMs with shape functions that reproduced polynomials of degree $k = 1$; it is not clear that the analysis can be extended to $k > 1$.

In this paper, we present a mathematical analysis of the effect of numerical integration on MM, where the quadrature is required to satisfy certain conditions that are different from those required in [4]. We also indicate how to obtain numerical quadrature schemes satisfying these conditions. Moreover, in contrast to [4], the

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[1] This research was partially supported by NSF Grant # 0611094.
[2] This research was partially supported by the NSF Grant # DMS-0610778.

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doi:10.1016/j.cma.2009.04.008

Please cite this article in press as: I. Babuška et al., Effect of numerical integration on meshless methods, Comput. Methods Appl. Mech. Engrg. (2009), doi:10.1016/j.cma.2009.04.008

analysis presented in this paper is valid for $k \geqslant 1$. We have shown in this paper that the energy norm of the error in the approximate solution obtained from MM with numerical quadrature (satisfying certain conditions) is $O(h^{k-1}(h+\eta))$, where η is a parameter related to the accuracy of the numerical quadrature and h is the standard discretization parameter. Thus MM does not yield optimal order of convergence for $\eta \neq O(h)$. Certainly if $\eta = O(h)$, we have the optimal order of convergence. It is important to note that the parameter η (see (3.10)) associated with the particular numerical integration used in the FEM, namely the Gauss rule, is $O(h)$. We mention that the numerical integration used in this paper yields a *non-symmetric* stiffness matrix. But this does not pose a serious problem since non-symmetric linear systems could be solved efficiently by iterative methods.

We address the application of MM on a second order Neumann boundary value problem in this paper. The outline of this paper is as follows: In Section 2, we present the preliminaries, a variational formulation based on Lagrange multipliers and the associated MM. In Section 3, we present a numerical quadrature scheme, together with associated assumptions on the scheme. We present our main results in Section 4, which are Theorems 4.1 and 4.2. In Section 5, we present a procedure that indicates how to obtain quadrature schemes satisfying an assumption given in Section 3. We also present numerical experiments in this section to illuminate our main results presented in Section 4. Some of these numerical experiments also indicate the necessity of one of the main assumptions on the quadrature given in Section 3. We provide a few remarks and a brief summary of the paper in Section 6.

2. Preliminaries and meshless method

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz continuous boundary $\Gamma \equiv \partial\Omega$. We denote the usual Sobolev space by $W^{m,p}(\Omega)$ with the norm and semi-norm, $\|u\|_{W^{m,p}(\Omega)}$ and $|u|_{W^{m,p}(\Omega)}$, respectively. We will consider only $p = 2$ and ∞ in this paper; $W^{m,2}(\Omega)$ will be denoted by $H^m(\Omega)$. Moreover, $\|u\|_{L_2(\Omega)}, \|u\|_{L_\infty(\Omega)}, \|u\|_{L_2(\Gamma)}$, and $\|u\|_{L_\infty(\Gamma)}$ will denote the usual norms on $L_2(\Omega)$, $L_\infty(\Omega), L_2(\Gamma)$, and $L_\infty(\Gamma)$, respectively.

Exact problem:

We consider the standard Neumann problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= g, \quad \text{on } \Gamma \equiv \partial\Omega, \end{aligned} \tag{2.1}$$

where $\frac{\partial}{\partial n}$ is the unit outward normal derivative to Γ and $f \in L_2(\Omega)$, $g \in L_2(\Gamma)$ satisfy the compatibility condition

$$\int_\Omega f(x)\,dx + \int_\Gamma g(s)\,ds = 0. \tag{2.2}$$

The associated variational formulation of (2.1) is given by

$$\begin{aligned} &\text{Find } u \in H^1(\Omega) \quad \text{satisfying,} \\ &B(u,v) = L(v), \quad \forall v \in H^1(\Omega), \end{aligned} \tag{2.3}$$

where

$$B(u,v) \equiv \int_\Omega \nabla u \cdot \nabla v\,dx \quad \text{and} \quad L(v) \equiv \int_\Omega f v\,dx + \int_\Gamma g v\,ds.$$

The compatibility condition (2.2) can be written as $L(1) = 0$. It is well known that the problem (2.3) has a unique solution up to a constant. A standard way of specifying a unique solution is to consider a linear functional $\Phi : L_2(\Omega) \to \mathbb{R}$ with $\Phi(1) > 0$ and seek the unique solution u satisfying $\Phi(u) = 0$. The functional $\Phi(u)$, for example, could be chosen to be $\Phi(u) = \frac{1}{|\Omega|}\int_\Omega u\,dx$ or $\Phi(u) = \int_\Omega \varphi u\,dx$, where $\varphi(x)$ is smooth. Let

$$H_\Phi = \left\{(v,\mu) \in H^1(\Omega) \times \mathbb{R} : \|(v,\mu)\|^2_{H_\Phi} \equiv |v|^2_{H^1(\Omega)} + |\Phi(v)|^2 + \mu^2 < \infty\right\}.$$

H_Φ is a Hilbert space and it is easy to show that there exist positive constants C_1, C_2, such that

$$C_1\|(v,\mu)\|^2_{H_\Phi} \leqslant \|v\|^2_{H^1(\Omega)} + \mu^2 \leqslant C_2\|(v,\mu)\|^2_{H_\Phi}, \quad \forall\ (v,\mu) \in H_\Phi. \tag{2.4}$$

We consider an alternate variational problem given by

$$\begin{aligned} &\text{Find } (u,\lambda) \in H_\Phi \quad \text{satisfying,} \\ &\mathscr{B}_\Phi(u,\lambda; v,\mu) = L(v), \quad \forall (v,\mu) \in H_\Phi, \end{aligned} \tag{2.5}$$

where

$$\mathscr{B}_\Phi(u,\lambda; v,\mu) \equiv B(u,v) + \lambda\Phi(v) + \mu\Phi(u),$$

and $B(u,v)$ and $L(v)$ are defined above.

Remark 2.1. We note that the problem (2.5) can equivalently be written as the system

$$\begin{aligned} &B(u,v) + \lambda\Phi(v) = L(v), \quad \forall v \in H^1(\Omega), \\ &\mu\Phi(u) = 0, \quad \forall \mu \in \mathbb{R}. \end{aligned}$$

The second equation gives the constraint $\Phi(u) = 0$. Moreover, it is well known that the first equation is the Euler–Lagrange equation for the constrained extremal problem

$$\min_{\substack{v \in H^1(\Omega) \\ \Phi(v)=0}} J(v),$$

where $J(v) = \frac{1}{2}B(v,v) - L(v).\lambda$ is the Lagrange multiplier and the problem (2.5) is known as the variational problem based on Lagrange multiplier.

To establish that the problem (2.5) has a unique solution, we present the following result.

Lemma 2.1

(a) *There is a constant $C > 0$ such that*

$$|\mathscr{B}_\Phi(u,\lambda; v,\mu)| \leqslant C\|(u,\lambda)\|_{H_\Phi}\|(v,\mu)\|_{H_\Phi}, \quad \forall (u,\lambda), (v,\mu) \in H_\Phi. \tag{2.6}$$

(b) *There exists $C > 0$ such that*

$$C < \inf_{(u,\lambda)\in H_\Phi} \sup_{(v,\mu)\in H_\Phi} \frac{\mathscr{B}_\Phi(u,\lambda; v,\mu)}{\|(u,\lambda)\|_{H_\Phi}\ \|(v,\mu)\|_{H_\Phi}}. \tag{2.7}$$

(c) *For any $(v,\mu) \in H_\Phi$ satisfying $\|(v,\mu)\|_{H_\Phi} \neq 0$,*

$$0 < \sup_{(u,\lambda)\in H_\Phi} |\mathscr{B}_\Phi(u,\lambda; v,\mu)|.$$

Proof

(a) This follows directly from the Cauchy-Schwartz inequality.

(b) We show that for a given $(u,\lambda) \in H_\Phi$, we can choose $(v,\mu) \in H_\Phi$ such that

$$\mathscr{B}_\Phi(u,\lambda; v,\mu) \geqslant C\|(u,\lambda)\|^2_{H_\Phi}$$

and

$$\|(v,\mu)\|_{H_\Phi} \leqslant C\|(u,\lambda)\|_{H_\Phi}.$$

We choose $v = u + \lambda$ and $\mu = -\lambda + \Phi(u)$. Then

$$\begin{aligned} \mathscr{B}_\Phi(u,\lambda; v,\mu) &= B(u,v) + \lambda\Phi(v) + \mu\Phi(u) \\ &= B(u,u+\lambda) + \lambda\Phi(u+\lambda) + [-\lambda + \Phi(u)]\Phi(u) \\ &= |u|^2_{1,\Omega} + \Phi(u)^2 + \lambda^2\Phi(1) \geqslant C\|(u,\lambda)\|^2_{H_\Phi}, \end{aligned}$$

and

$$\begin{aligned}\|(v,\mu)\|_{H_\Phi}^2 &= |v|_{1,\Omega}^2 + \Phi(v)^2 + \mu^2\\ &= |u+\lambda|_{1,\Omega}^2 + [\Phi(u+\lambda)]^2 + [-\lambda + \Phi(u)]^2\\ &= |u|_{1,\Omega}^2 + 2\Phi(u)^2 + \lambda^2\Phi(1)^2 + 2\Phi(1)\,\lambda\,\Phi(u)\\ &\quad - 2\lambda\,\Phi(u) + \lambda^2\\ &\leqslant C[|u|_{1,\Omega}^2 + \Phi(u)^2 + \lambda^2] = C\|(u,\lambda)\|_{H_\Phi}^2.\end{aligned}$$

Estimate (2.7) follows from these two inequalities.

(c) For a given $(v,\mu)\in H_\Phi$, we choose $u = v+\mu$ and $\lambda = -\mu + \Phi(v)$. Using a similar calculation as used in the first part of the proof of (b), we get the desired result. □

It now follows from Theorem 5.2.1 in [1] that the problem (2.5) has a unique solution.

Remark 2.2. We note that the problem (2.5) has a unique solution (u,λ) for any $f\in L_2(\Omega)$ and $g\in L_2(\Gamma)$. Let the linear functional $\Phi(v)$ be given by $\Phi(v)\equiv\int_\Omega \varphi\, v\,dx$, where $\varphi(x)$ is smooth. Then, if u is smooth, it can be shown that u is the unique (strong) solution of the Neumann problem

$$\begin{aligned} -\Delta u &= f - \frac{L(1)}{\int_\Omega \varphi\,dx}\varphi \quad \text{in } \Omega,\\ \frac{\partial u}{\partial n} &= g, \quad \text{on } \Gamma \end{aligned} \tag{2.8}$$

with $\Phi(u)=0$ (see [6]). It can also be shown that $\lambda = L(1)/\Phi(1)$. If f and g satisfy the compatibility condition (2.2), i.e., $L(1)=0$, then it is clear from (2.8) that u is the solution of the original Neumann problem (2.1) with $\Phi(u)=0$.

Remark 2.3. Consider the variational problem (2.5), where we assumed that $L(1)=0$. Substituting $v=1$ in (2.5), it is easy to see that $\lambda=0$ and we can also show that the problem (2.5) is equivalent to the problem

$$\begin{aligned}&\text{Find } u\in H^1(\Omega) \quad \text{such that}\\ &B(u,v) = L(v) \quad \text{and} \quad \Phi(u)=0, \quad \forall v\in H^1(\Omega).\end{aligned} \tag{2.9}$$

Remark 2.4. The variational formulation (2.5) of the Neumann problem (2.1) and (2.2) is different than the standard variational formulation used in the literature [7]. We note that small perturbations in the input data (e.g., caused by the round-off error) or the quadrature error will disturb the compatibility condition (2.2). It is well known that the compatibility condition is necessary for the existence of the solution of the Neumann problem, and thus the standard variational formulation of the Neumann problem is not well-posed without a constraint on the perturbation. In contrast, the formulation (2.5) is well-posed without any constraint on the perturbation of data. We further note that there is obvious freedom in the selection of Φ.

Discretization:

In order to discretize the variational problem (2.5) by a meshless method, we consider $V_h\subset H^1(\Omega)$, a one-parameter family of finite dimensional spaces, given by

$$V_h = \text{span}\left\{\phi_j^h : j\in N_h\right\}; N_h \quad \text{is an index set}.$$

The shape functions $\{\phi_j^h(x)\}_{j\in N_h}$ are linearly independent. Moreover, ϕ_j^h's have compact support and (in a meshless method) *their construction either does not depend, or depends only minimally, on a mesh*. We let $\omega_j^h\subset\Omega$ be the interior of the supp ϕ_j^h. We assume that ω_j^h is star-shaped with respect to a ball $o_j^h\subset\omega_j^h$ and there exists a constant $C>0$ such that

$$\frac{\text{diam}(\omega_j^h)}{\text{diam}(o_j^h)} \geqslant C, \quad \forall j\in N_h.$$

For the definition of star-shaped domains with respect to a ball, we refer to [7].

Often, a shape function $\phi_i^h(x)$ is associated with a *particle* $x_i^h\in\mathbb{R}^d$ and it is assumed that the particles are distinct i.e., $x_i^h\neq x_j^h$ if $i\neq j$. We note that when $\overline{\omega}_i^h\cap\Gamma=\emptyset$, then the associated particle $x_i^h\in\omega_i^h\subset\Omega$. But when $\overline{\omega}_i^h\cap\Gamma\neq\emptyset$, then the associated particle x_i^h could be outside Ω. We also divide the set N_h into two sets, namely,

$$N_h'' = \{i\in N_h : \overline{\omega}_i^h\subset\Omega\}, \tag{2.10}$$

$$N_h' = \{i\in N_h : \overline{\omega}_i^h\cap\Gamma\neq\emptyset\}. \tag{2.11}$$

We note that $N_h = N_h''\cup N_h'$ and $N_h''\cap N_h'=\emptyset$. We set $|N_h| = \text{card}N_h$.

We now make the following assumptions on the subspace V_h.

A1: (finite overlap) For $i\in N_h$, let S_i be the set of indices j such that $\omega_i^h\cap\omega_j^h\neq\emptyset$. There is a constant κ, independent of i and h, such that

$$\text{card}\,S_i\leqslant\kappa. \tag{2.12}$$

A2: There are positive constants C, C_1, and C_2, independent of i and h, such that

$$\|D^\alpha\phi_i^h\|_{L_\infty(\Omega)}\leqslant Ch^{-|\alpha|},\ 0\leqslant|\alpha|\leqslant q \quad \text{for some } q\geqslant 1, \quad (\alpha \text{ is a multi-index}), \tag{2.13}$$

$$C_1h^d\leqslant|\omega_i^h|\leqslant C_2h^d \quad \text{and} \quad C_1h^{d-1}\leqslant|\overline{\omega}_i^h\cap\Gamma|\leqslant C_2h^{d-1}, \tag{2.14}$$

where $|\omega_i^h|$ is the "area" of ω_i^h in $\mathbb{R}^d$ and $|\overline{\omega}_i^h\cap\Gamma|$ is the "length" of $\overline{\omega}_i^h\cap\Gamma$ in $\mathbb{R}^{d-1}$.

A3: There are positive constants C_1 and C_2, independent of h and i, such that

$$C_1\leqslant\frac{\text{diam}(\omega_i^h)}{h}\leqslant C_2. \tag{2.15}$$

A4: There are positive constants C_1 and C_2, independent of h and i, such that

$$C_1\|v\|_{L_2(\Omega)}^2\leqslant h^d\sum_{i\in N_h}v_i^2\leqslant C_2\|v\|_{L_2(\Omega)}^2, \tag{2.16}$$

$$C_1\|v\|_{L_2(\Gamma)}^2\leqslant h^{d-1}\sum_{i\in N_h'}v_i^2\leqslant C_2\|v\|_{L_2(\Gamma)}^2, \tag{2.17}$$

$$C_1|v|_{H^1(\omega_i)}^2\leqslant h^{d-2}\sum_{j\in S_i}(v_j-v_i)^2\leqslant C_2|v|_{H^1(\omega_i)}^2, \quad \forall i\in N_h, \tag{2.18}$$

for all $v=\sum_{i\in N_h}v_i\phi_i^h\in V_h$.

A5: The shape functions *reproduce polynomials of degree* k, i.e.,

$$\sum_{i\in N_h}p(x_i^h)\phi_i^h(x) = p(x), \quad \forall p\in\mathscr{P}^k(\Omega) \quad \text{and} \quad x\in\Omega, \tag{2.19}$$

where $\mathscr{P}^k$ is the space of polynomials of degree k.

Remark 2.5. The assumption **A3** implies the first statement of (2.14). However, we stated them separately since we have used these statements in this paper. We next note that the inequality (2.16) in **A4** implies a strengthened uniform version of linear independence of the shape functions $\{\phi_i^h\}$. Also note that (2.16) and (2.17) in assumption **A4** imply that

$$|N_h| \approx Ch^{-d}, \ |N''_h| \approx Ch^{-d}, \quad \text{and} \quad |N'_h| \approx Ch^{-(d-1)} \tag{2.20}$$

provided the function $v = 1 \in V_h$. Here the notation $A_h \approx B_h$ means that there are constants C_1, C_2, independent of h such that $C_1 \leqslant |A_h|/|B_h| \leqslant C_2$. It is clear from assumption **A5** (take $p(x) = 1$ in (2.19)) that $1 \in V_h$ and therefore $\{\phi_i^h\}$ form a partition of unity. We mention that the particles $\{x_i^h\}$ are used in the construction of shape functions satisfying (2.19) (see [18]). We further note that it is possible to prove the inequalities (2.16)–(2.18) in certain situations for special distributions of the particles $\{x_i^h\}$; also see Remark 4.3 in [4]. These proofs require the assumption (2.15).

Examples of subspaces V_h satisfying these assumptions can be found in [2,17,20]. We mention that it is easy to construct smooth shape functions $\{\phi_i^h\}$, e.g., RKP shape functions with respect to a smooth weight function (see [19,17,18]). We assume that $\phi_i^h \in C^{k+1}(\overline{\Omega})$ for all $i \in N_h$.

In the rest of the paper, we will write $x_j^h, \omega_j^h, \phi_j^h$ as x_j, ω_j, ϕ_j, respectively, for notational simplicity, with the understanding that they depend on h.

We will use a Lagrange multiplier method to determine a unique approximate solution for problem (2.5). To this end, we define a linear functional Ψ on V_h by

$$\Psi(v_h) = \frac{\Phi(1)}{|N_h|} \sum_{i \in N_h} v_i, \quad \forall v_h = \sum_{i \in N_h} v_i \phi_i \in V_h.$$

Ψ is different from Φ, but note that $\Psi(1) = \Phi(1)$. Also Ψ is a bounded linear functional since

$$|\Psi(v_h)| \leqslant \frac{\Phi(1)}{|N_h|} \left(\sum_{i \in N_h} 1\right)^{\frac{1}{2}} \left(\sum_{i \in N_h} v_i^2\right)^{\frac{1}{2}} \leqslant Ch^d h^{-\frac{d}{2}} h^{-\frac{d}{2}} \|v_h\|_{L_2(\Omega)} = C\|v_h\|_{L_2(\Omega)}, \tag{2.21}$$

where the last inequality is obtained using (2.16) and (2.20). We consider

$$V_\Psi^h \equiv \left\{(v_h, \mu) \in V_h \times \mathbb{R} : \|(v_h, \mu)\|^2_{V_\Psi^h} \equiv |v_h|^2_{H^1(\Omega)} + |\Psi(v_h)|^2 + \mu^2 < \infty\right\}.$$

V_Ψ^h is a Hilbert space and we can show that there are constants C_1, C_2, independent of h, such that

$$C_1 \|(v_h, \mu)\|^2_{V_\Psi^h} \leqslant \|v_h\|^2_{H_1} + \mu^2 \leqslant C_2 \|(v_h, \mu)\|^2_{V_\Psi^h}, \forall (v_h, \mu) \in V_\Psi^h. \tag{2.22}$$

Therefore from (2.4) we see that the norms $\|(\cdot,\cdot)\|_{V_\Psi^h}$ and $\|(\cdot,\cdot)\|_{H_\Phi}$ are equivalent on V_Ψ^h and the associated constants are independent of h. We note that the linear functional Ψ is not well defined on $H^1(\Omega)$, but is well defined on V_h.

Meshless method:

A meshless method to approximate the solution of (2.5) is a Galerkin method

$$\begin{aligned} &\text{Find } (u_h, \lambda_h) \in V_\Psi^h \quad \text{satisfying} \\ &\mathscr{B}_\Psi(u_h, \lambda_h; v, \mu) = L(v), \quad \forall (v, \mu) \in V_\Psi^h, \end{aligned} \tag{2.23}$$

where

$$\mathscr{B}_\Psi(u_h, \lambda_h; v, \mu) \equiv B(u_h, v) + \lambda_h \Psi(v) + \mu \Psi(u_h).$$

We will give a rationale for using Ψ (in place of Φ) in (2.23) later in Remark 3.1.

Let $z \in C(\overline{\Omega})$. We define $\mathscr{I}_h z \in V_h$, called the V_h-**interpolant** of z, as

$$\mathscr{I}_h z \equiv \sum_{i \in N_h} z(x_i)\phi_i(x). \tag{2.24}$$

Strictly speaking, $\mathscr{I}_h z$ is a quasi-interpolant of z, since $\mathscr{I}_h z(x_j) \neq z(x_j)$. We further note that the function z has to be suitably extended outside Ω in order to define $\mathscr{I}_h z$, since some of the particles x_i may be outside Ω. The extension of z to define $\mathscr{I}_h z$ has been discussed in [3].

In the following result, we present the interpolation error estimate.

Lemma 2.2. *Let $z \in W^{k+1,p}(\Omega) \cap C(\overline{\Omega}), p = 2, \infty$. Then there exists a constant C, independent of z and h such that*

$$\|z - \mathscr{I}_h z\|_{W^{s,p}(\Omega)} \leqslant Ch^{k+1-s}|z|_{W^{k+1,p}(\Omega)}. \tag{2.25}$$

The proof of this theorem can be found in [17,2].

Remark 2.6. It is important to note that the proof of Lemma 2.2 needs an additional assumption. We assume that for each $i \in N_h$, there is a ball $\mathbb{B}_i$ of diameter ρh such that

$$\bigcup_{j \in S_i} \omega_j \subset \mathbb{B}_i,$$

where $\rho \geqslant 1$ is independent of i. We further note that the proof also depends on the assumption **A5**, which in turn imposes certain restrictions on the distribution of the particles $\{x_i\}$, which we do not elaborate here.

To address the existence and uniqueness of the solution of the problem (2.23), we state the following result:

Lemma 2.3

(a) *There is a constant $C > 0$, independent of h, such that*

$$|\mathscr{B}_\Psi(w, v; v, \mu)| \leqslant C\|(w, v)\|_{V_\Psi^h}\|(v, \mu)\|_{V_\Psi^h}, \quad \forall\ (w, v), (v, \mu) \in V_\Psi^h.$$

(b) *There exists $C > 0$, independent of h, such that*

$$C < \inf_{(w,v) \in V_\Psi^h} \sup_{(v,\mu) \in V_\Psi^h} \frac{\mathscr{B}_\Psi(w, v; v, \mu)}{\|(w, v)\|_{V_\Psi^h}\|(v, \mu)\|_{V_\Psi^h}}.$$

(c) *For any $(v, \mu) \in V_\Psi^h$ satisfying $\|(v, \mu)\|_{V_\Psi^h} \neq 0$,*

$$0 < \sup_{(w,v) \in V_\Psi^h} \mathscr{B}_\Psi(w, v; v, \mu).$$

The proof of this result depends on the fact that V_h contains constants (because of assumption **A5**) and follows the same arguments used in the proof of Lemma 2.1.

Remark 2.7. Since V_h contains constants, by considering $v_h = 1$ in the variational problem (2.23) it is clear that $\lambda_h = 0$ and we can also show that the problem (2.23) is equivalent to

$$\begin{aligned} &\text{Find } u_h \in V_h \quad \text{such that} \\ &B(u_h, v_h) = L(v_h) \quad \text{and} \quad \Psi(u_h) = 0, \quad \forall v_h \in V_h. \end{aligned} \tag{2.26}$$

We note that the constraint $\Psi(u_h) = 0$ gives a non-singular stiffness matrix (which will otherwise be singular). This feature, possibly with a different choice of Ψ, is always incorporated in a standard FEM code.

Now it is immediate from (2.9) and (2.26),

$$B(u - u_h, v_h) = 0, \quad \forall v_h \in V_h$$

and therefore,

$$|u - u_h|_{H^1(\Omega)} \leqslant \inf_{v_h \in V_h} |u - v_h|_{H^1(\Omega)} \leqslant Ch^k |u|_{H^{k+1}(\Omega)}. \tag{2.27}$$

Thus u_h converges to u only in the "energy norm", i.e., the H^1-seminorm. Moreover, using (2.25), we also obtain

$$|u_h - \mathscr{I}_h u|_{H^1(\Omega)} \leqslant |u_h - u|_{H^1(\Omega)} + |u - \mathscr{I}_h u|_{H^1(\Omega)} \leqslant Ch^k |u|_{H^{k+1}(\Omega)}, \tag{2.28}$$

which will be used later in the paper.

Please cite this article in press as: I. Babuška et al., Effect of numerical integration on meshless methods, Comput. Methods Appl. Mech. Engrg. (2009), doi:10.1016/j.cma.2009.04.008

3. Numerical integration in meshless method

To motivate the quadrature in the meshless method, we first look at the problem (2.23) in detail. We write u_h, the solution of (2.23), as $u_h = \sum_{j\in N_h} c_j\phi_j$. Then the problem (2.23) can be written as

$$\sum_{j\in N_h}\gamma_{ij} + \lambda_h\Psi(\phi_i) + \frac{\mu\Phi(1)}{|N_h|}\sum_{j\in N_h} c_j = l_i, \quad \text{for } i\in N_h, \forall\mu\in\mathbb{R},$$

where

$$\gamma_{ij} = B(\phi_j,\phi_i) = \int_\Omega \nabla\phi_j\cdot\nabla\phi_i\,dx = \int_{\omega_j\cap\omega_i}\nabla\phi_j\cdot\nabla\phi_i\,dx = \int_{\omega_i}\nabla\phi_j\cdot\nabla\phi_i\,dx$$

and

$$l_i = L(\phi_i) = \int_\Omega f\phi_i\,dx + \int_\Gamma g\phi_i\,ds = \int_{\omega_i} f\phi_i\,dx + \int_{\Gamma\cap\overline{\omega}_i} g\phi_i\,ds \equiv f_i + g_i. \tag{3.1}$$

The integrals γ_{ij}, f_i, and g_i are computed using quadrature. We define

$$\gamma^*_{ij} = ⨍_{\omega_i}\nabla\phi_j\cdot\nabla\phi_i\,dx \tag{3.2}$$

and

$$l^*_i = ⨍_{\omega_i} f\phi_i\,dx + ⨍_{\Gamma\cap\overline{\omega}_i} g\phi_i\,ds \equiv f^*_i + g^*_i, \tag{3.3}$$

where ⨍ represents the numerically computed integral $\int$. *We note that the matrix* $\{\gamma^*_{ij}\}$ *is not symmetric, since*

$$\gamma^*_{ij} = ⨍_{\omega_i}\nabla\phi_j\cdot\nabla\phi_i\,dx \neq ⨍_{\omega_j}\nabla\phi_i\cdot\nabla\phi_j\,dx = \gamma^*_{ji}.$$

We next note that for $v = \sum_{j\in N_h} v_j\phi_j$ and $w = \sum_{i\in N_h} w_i\phi_i$ in V_h, we have

$$B(v,w) = \sum_{i,j\in N_h}\gamma_{ij}v_jw_i$$

$$\text{and}\quad L(v) = \sum_{i\in N_h} f_iv_i + \sum_{i\in N'_h} g_iv_i.$$

So we naturally define

$$B^*(v,w) = \sum_{i,j\in N_h}\gamma^*_{ij}v_jw_i \quad \text{and}\quad L^*(v) = \sum_{i\in N_h} f^*_iv_i + \sum_{i\in N'_h} g^*_iv_i. \tag{3.4}$$

Under this definition, the form $B^*(\cdot,\cdot)$ is bilinear on $V_h\times V_h$ and $L^*(\cdot)$ is linear on V_h. Since $\{\gamma^*_{ij}\}$ is not symmetric, it is clear that $B^*(v,w)$ is also not symmetric. Moreover, it can be easily shown that

$$\sum_{j\in N_h}\gamma^*_{ij} = 0, \quad \forall\, i\in N_h, \tag{3.5}$$

i.e., the "row-sum" of the matrix $\{\gamma^*_{ij}\}$ is 0, which implies that

$$B^*(1,w) = 0, \quad \forall w\in V_h. \tag{3.6}$$

But, in general, the "column-sum" of $\{\gamma^*_{ij}\}$ is not 0 and there is $v\in V_h$ such that $B^*(v,1)\neq 0$. We also observe that

$$L^*(1) = \sum_{i\in N_h} f^*_i + \sum_{i\in N'_h} g^*_i \neq 0. \tag{3.7}$$

It is important to note that several other approaches for performing numerical integration have been proposed in the literature to approximate γ_{ij}. We mention some of them below:

(i) The domain Ω is partitioned into small and appropriately selected subdomains such that each ω_i and $\omega_i\cap\omega_j$ are also unions of these subdomains [16]. Numerical integration is then performed on each of these subdomains. The choice of subdomains in this approach allows the numerical integration of only smooth functions. The computed stiffness matrix $\{\gamma^*_{ij}\}$ is also symmetric, but partitioning process could be 'expensive' when ω_i's are not simplices.

(ii) In a procedure proposed in [11,12,10], the domain Ω is partitioned into subdomains such that each ω_i and $\omega_i\cap\omega_j$ are also unions of these subdomains. Moreover, these subdomains contain exactly one particle x_j. The main feature of this procedure is that the numerical integration is performed only on the boundary of the subdomains. Also the matrix $\{\gamma^*_{ij}\}$ is symmetric. The effectiveness of this approach for the lowest order, i.e., when $k=1$ in (2.19), was shown in [11,12]; the higher order methods, i.e., $k\geqslant 2$ in (2.19), were addressed in [10].

(iii) The element γ_{ij} of the stiffness matrix could be approximated by performing numerical integration on $\omega_i\cap\omega_j$. This approach was used in [14,15], where the domains ω_i's were considered as spheres. Consequently, the domains $\omega_i\cap\omega_j$ were "lens-shaped" and special quadrature formulae were developed to numerically integrate over such lens-shaped domains. The matrix $\{\gamma^*_{ij}\}$, obtained from this procedure, is symmetric, but the main drawback of is that the matrix $\{\gamma^*_{ij}\}$ does not satisfy the zero row sum condition (3.5).

(iv) Numerical integration on $\omega_i\cap\omega_j$, together with a simple "correction", to approximate γ_{ij} was suggested in [4]. This correction ensured the zero row sum condition (3.5) for the matrix $\{\gamma^*_{ij}\}$; the matrix was also symmetric. The mathematical analysis presented in this paper required a certain assumption on the discretization, which is not easy to check. Also the analysis is valid for shape functions satisfying $k=1$ in (2.19); it could not be generalized for $k\geqslant 2$.We note that rigorous mathematical error analysis is not available for the procedures described in (i)–(iii). In this paper, we considered numerical integration on ω_i (not on $\omega_i\cap\omega_j$) of the form (3.2) and (3.3) to approximate γ_{ij} and, unlike [4], our error analysis is valid for all $k\geqslant 1$ in (2.19).

The meshless method with quadrature to approximate the solution u of the problem (2.5) is given by

$$\text{Find } (u^*_h,\lambda^*_h)\in V^h_\Psi \quad \text{satisfying} \quad \mathscr{B}^*_\Psi(u^*_h,\lambda^*_h; v,\mu) = L^*(v), \quad \forall(v,\mu)\in V^h_\Psi, \tag{3.8}$$

where

$$\mathscr{B}^*_\Psi(w,\nu;v,\mu) \equiv B^*(w,v) + \nu\Psi(v) + \mu\Psi(w) \tag{3.9}$$

and $B^*(\cdot,\cdot)$ and $L^*(\cdot)$ are as defined in (3.4). We refer to u^*_h as the quadrature approximation to u. It is clear from (3.7) that the compatibility condition is not satisfied.

Remark 3.1. We note that the main reason for using $\Psi(\cdot)$ in (2.23) and (3.8) instead of $\Phi(\cdot)$, is that the numerical approximation of the linear functional $\Phi(\cdot)$ is not required in (3.8).

We assume that the numerical quadrature satisfies the conditions described below:

QA1. There exist positive constants η and τ, small enough and independent of i and h, such that

$$\left|\int_{\omega_i}\varrho\,dx - ⨍_{\omega_i}\varrho\,dx\right| \leqslant \eta\,|\omega_i|\,\|\varrho\|_{L_\infty(\omega_i)} \tag{3.10}$$

and

$$\left|\int_{\partial\omega_i\cap\Gamma}\vartheta\,ds - ⨍_{\partial\omega_i\cap\Gamma}\vartheta\,ds\right| \leqslant \tau\,|\partial\omega_i\cap\Gamma|\,\|\vartheta\|_{L_\infty(\partial\omega_i\cap\Gamma)} \tag{3.11}$$

for a class of functions $\varrho \in W^{m_1,\infty}(\omega_i)$ and $\vartheta \in W^{m_2,\infty}(\partial\omega_i\cap\Gamma)$ satisfying

$$\|D^\alpha\varrho\|_{L_\infty(\omega_i)} \leqslant C[\mathrm{diam}(\omega_i)]^{-|\alpha|}\|\varrho\|_{L_\infty(\omega_i)}, \quad |\alpha| \leqslant m_1 \tag{3.12}$$

and

$$\|D^\alpha\vartheta\|_{L_\infty(\partial\omega_i\cap\Gamma)} \leqslant C[\mathrm{diam}(\omega_i)]^{-|\alpha|}\|\vartheta\|_{L_\infty(\partial\omega_i\cap\Gamma)}, \quad |\alpha| \leqslant m_2, \tag{3.13}$$

where $C > 0$ is independent of i and $m_1, m_2 \geqslant 1$ may depend on the numerical quadrature as well as on q in assumption **A2**.

QA2.

(a) There is a constant $C > 0$, independent of h, such that

$$|\mathscr{B}^*_\Psi(w,v;\upsilon,\mu)| \leqslant C\|(w,v)\|_{V^h_\Psi}\|(\upsilon,\mu)\|_{V^h_\Psi}, \quad \forall\ (w,v),(\upsilon,\mu) \in V^h_\Psi. \tag{3.14}$$

(b) There exists $C > 0$, independent of h, such that

$$C < \inf_{(w,v)\in V^h_\Psi}\ \sup_{(\upsilon,\mu)\in V^h_\Psi} \frac{\mathscr{B}^*_\Psi(w,v;\upsilon,\mu)}{\|(w,v)\|_{V^h_\Psi}\|(\upsilon,\mu)\|_{V^h_\Psi}}. \tag{3.15}$$

(c) For any $(\upsilon,\mu) \in V^h_\Psi$ satisfying $\|(\upsilon,\mu)\|_{V^h_\Psi} \neq 0$,

$$0 < \sup_{(w,v)\in V^h_\Psi} \mathscr{B}^*_\Psi(w,v;\upsilon,\mu). \tag{3.16}$$

QA3. For each $i \in N_h$, let $G^*_i : C^2(\overline{\omega}_i) \to \mathbb{R}$ be a linear functional given by

$$G^*_i(\upsilon) = ⨍_{\omega_i}\nabla\upsilon\cdot\nabla\phi_i\,dx + ⨍_{\omega_i}\Delta\upsilon\,\phi_i\,dx - ⨍_{\partial\omega_i\cap\Gamma}\nabla\upsilon\cdot\vec{n}\,\phi_i\,ds, \tag{3.17}$$

where $\vec{n}$ is the unit outward normal to $\partial\omega_i\cap\Gamma$. We assume that

$$G^*_i(p) = 0, \quad \forall p \in \mathscr{P}^k \quad \text{and} \quad \forall i \in N_h, \tag{3.18}$$

where $\mathscr{P}^k$ is the space of polynomials of degree k.

We first note that the assumption **QA2**, i.e., (3.14), (3.15), and (3.16) ensures that the problem (3.8) has a unique solution. In the following lemma, we show that (3.14)–(3.16) hold under a somewhat restricted condition on the parameter η.

Lemma 3.1. *Suppose there is a positive constant C such that the quadrature satisfies (3.10) with $\eta \leqslant Ch$. Then $\mathscr{B}^*_\Psi(w,v;\upsilon,\mu)$ is bounded, and for C small enough, $\mathscr{B}^*_\Psi(w,v;\upsilon,\mu)$ satisfies the inf–sup conditions, i.e., (3.15) and (3.16) are satisfied.*

Proof. Let

$$w = \sum_{i\in N_h} w_i\phi_i, \quad \upsilon = \sum_{i\in N_h}\upsilon_i\phi_i \in V_h.$$

We first estimate $|B(w,\upsilon) - B^*(w,\upsilon)|$.

Recalling the definition of η in (3.10) and using (3.6) and (2.18), we have for $i \in N_h$,

$$\begin{aligned}
&|B(w,\phi_i) - B^*(w,\phi_i)| \\
&= \left|\int_{\omega_i}\nabla w\cdot\nabla\phi_i\,dx - ⨍_{\omega_i}\nabla w\cdot\nabla\phi_i\,dx\right| \\
&= \left|\int_{\omega_i}\nabla(w-w_i)\cdot\nabla\phi_i\,dx - ⨍_{\omega_i}\nabla(w-w_i)\cdot\nabla\phi_i\,dx\right| \\
&\leqslant \sum_{j\in S_i}|w_j - w_i|\left|\int_{\omega_i}\nabla\phi_j\cdot\nabla\phi_i\,dx - ⨍_{\omega_i}\nabla\phi_j\cdot\nabla\phi_i\,dx\right| \\
&\leqslant \eta\,|\omega_i|\,\|\nabla\phi_j\cdot\nabla\phi_i\|_{L_\infty(\omega_i)}\left(\sum_{j\in S_i}|w_j - w_i|^2\right)^{\frac{1}{2}}\sqrt{\kappa} \\
&\leqslant C_2\eta h^d h^{-2} h^{-\frac{d-2}{2}}|w|_{H^1(\omega_i)}\sqrt{\kappa} \leqslant C\eta h^{\frac{d}{2}-1}|w|_{H^1(\omega_i)},
\end{aligned} \tag{3.19}$$

where we used (2.13) with $|\alpha| = 1$. Therefore, squaring both sides of the above inequality and summing over $i \in N_h$, we get

$$\sum_{i\in N_h}[B(w,\phi_i) - B^*(w,\phi_i)]^2 \leqslant C\eta^2h^{d-2}\sum_{i\in N_h}|w|^2_{H^1(\omega_i)} \leqslant C\eta^2h^{d-2}|w|^2_{H^1(\Omega)}.$$

Thus, recalling that $\upsilon = \sum_{i\in N_h}\upsilon_i\phi_i$ and using (2.16), we get

$$\begin{aligned}
|B(w,\upsilon) - B^*(w,\upsilon)| &= \left|\sum_{i\in N_h}\upsilon_i[B(w,\phi_i) - B^*(w,\phi_i)]\right| \\
&\leqslant \left(\sum_{i\in N_h}\upsilon_i^2\right)^{1/2}\left[\sum_{i\in N_h}[B(w,\phi_i) - B^*(w,\phi_i)]^2\right]^{1/2} \\
&= C\eta h^{\frac{d}{2}-1}|w|_{H^1(\Omega)}\left(\sum_{i\in N_h}\upsilon_i^2\right)^{1/2} \\
&\leqslant C\eta h^{-1}|w|_{H^1(\Omega)}\|\upsilon\|_{L_2(\Omega)}.
\end{aligned} \tag{3.20}$$

We now prove the boundedness of $\mathscr{B}^*_\Psi$, i.e., (3.14). From the definition (3.9) of $\mathscr{B}^*_\Psi(w,v;\upsilon,\mu)$, we get

$$\mathscr{B}^*_\Psi(w,v;\upsilon,\mu) = \mathscr{B}_\Psi(w,v;\upsilon,\mu) - [B(w,\upsilon) - B^*(w,\upsilon)]. \tag{3.21}$$

Therefore, using (3.20) and part (a) of Lemma 2.3, we get

$$\begin{aligned}
|\mathscr{B}^*_\Psi(w,v;\upsilon,\mu)| &\leqslant |\mathscr{B}_\Psi(w,v;\upsilon,\mu)| + |B(w,\upsilon) - B^*(w,\upsilon)| \\
&\leqslant C_1\|(w,v)\|_{V^h_\Psi}\|(\upsilon,\mu)\|_{V^h_\Psi} + C_2\eta h^{-1}|w|_{H^1(\Omega)}\|\upsilon\|_{L_2(\Omega)} \\
&\leqslant C(1+\eta h^{-1})\|(w,v)\|_{V^h_\Psi}\|(\upsilon,\mu)\|_{V^h_\Psi}.
\end{aligned}$$

Thus by taking $\eta \leqslant Ch$ (C does not have to be small enough) in the above inequality, we get (3.14).

We now prove the inf–sup condition (3.15). For a given $(w,v) \in V^h_\Psi$, we choose $\upsilon = w + v \in V_h$ and $\mu = -v + \Psi(w)$. It can be shown following the proof of (2.7) that

$$\mathscr{B}_\Psi(w,v;\upsilon,\mu) \geqslant C_1\|(w,v)\|^2_{V^h_\Psi} \text{ and } \|(\upsilon,\mu)\|_{V^h_\Psi} \leqslant C\|(w,v)\|_{V^h_\Psi}. \tag{3.22}$$

Therefore, from (3.21),

$$\mathscr{B}^*_\Psi(w,v;\upsilon,\mu) \geqslant C_1\|(w,v)\|^2_{V^h_\Psi} - |B(w,\upsilon) - B^*(w,\upsilon)|. \tag{3.23}$$

Since $\upsilon = w + v$, υ could be written as

$$\upsilon = \sum_{i\in N_h}\upsilon_i\phi_i, \quad \text{where } \upsilon_i = w_i + v.$$

Therefore, from (2.16) and (2.20)

$$\begin{aligned}
\|\upsilon\|^2_{L_2(\Omega)} &\leqslant Ch^d\sum_{i\in N_h}\upsilon_i^2 = Ch^d\sum_{i\in N_h}(w_i+v)^2 \leqslant Ch^d\sum_{i\in N_h}(w_i^2+v^2) \\
&= Ch^d\sum_{i\in N_h}w_i^2 + Ch^d\sum_{i\in N_h}v^2 \leqslant C[\|w\|^2_{L_2(\Omega)} + v^2].
\end{aligned} \tag{3.24}$$

Using the above in (3.20), we get

$$\begin{aligned}
|B(w,\upsilon) - B^*(w,\upsilon)| &\leqslant C\eta h^{-1}|w|_{H^1(\Omega)}[\|w\|_{L_2(\Omega)} + |v|] \\
&\leqslant C\eta h^{-1}[|w|^2_{H^1(\Omega)} + \|w\|^2_{L_2(\Omega)} + |v|^2] \\
&\leqslant C_2\eta h^{-1}[|w|^2_{H^1(\Omega)} + |\Psi(w)|^2 + |v|^2] \\
&= C_2\eta h^{-1}\|(w,v)\|^2_{V^h_\Psi}.
\end{aligned} \tag{3.25}$$

Therefore, from (3.23), we have

$$\mathscr{B}^*_\Psi(w,v;\upsilon,\mu) \geqslant [C_1 - C_2\eta h^{-1}]\|(w,v)\|^2_{V^h_\Psi}.$$

Finally, considering ηh^{-1} small enough such that $[C_1 - C_2\eta h^{-1}] \geqslant C > 0$, we get

$$\mathscr{B}^*_\Psi(w,v;v,\mu) \geqslant C\|(w,v)\|^2_{V^h_\Psi}.$$

We have already seen from (3.22) that

$$\|(v,\mu)\|_{V^h_\Psi} \leqslant C\|(w,v)\|_{V^h_\Psi}.$$

Thus we proved the inf–sup condition (3.15). The proof of (3.16) is similar to Lemma 2.1(c) and we do not provide it here. □

Remark 3.2. We note that Lemma 3.1 was proved under a restrictive condition on η, namely, we required that $\eta \leqslant Ch$, with C sufficiently small. Computations suggest that the condition $\eta = O(h)$ is not necessary for the existence of a unique solution of the problem (3.8) (η sufficiently small, independent of h, is sufficient). We will further comment on the dependence of η on h later in this paper.

Remark 3.3. We will indicate in this remark that it is possible to choose a quadrature rule that yields a small η in (3.10) in the assumption **QA1**. We consider the set

$$\hat{\omega}_i = \{\xi \in \mathbb{R}^d : \xi = x/\mathrm{diam}(\omega_i),\ \text{where } x \in \omega_i\}.$$

Clearly, $\mathrm{diam}(\hat{\omega}_i) = 1$. For $\varrho \in W^{m_1,\infty}(\omega_i)$ satisfying (3.12), we define $\hat{\varrho}(\xi) \equiv \varrho(x) = \varrho(\xi \mathrm{diam}(\omega_i))$. Then it is easy to show that

$$\|\hat{\varrho}\|_{L_\infty(\hat{\omega}_i)} = \|\varrho\|_{L_\infty(\omega_i)} \quad \text{and} \quad \|D^\alpha\hat{\varrho}\|_{L_\infty(\hat{\omega}_i)} \leqslant C\|\hat{\varrho}\|_{L_\infty(\hat{\omega}_i)}, \quad |\alpha| \leqslant m_1. \tag{3.26}$$

We now consider a n_i-point quadrature rule on $\hat{\omega}_i$ such that

$$\left|\int_{\hat{\omega}_i} \hat{\varrho}(\xi)\,d\xi - ⨍_{\hat{\omega}_i} \hat{\varrho}(\xi)\,d\xi\right| \leqslant \hat{\eta}\|D^\alpha\hat{\varrho}\|_{L_\infty(\hat{\omega}_i)}, \quad |\alpha| = m_1, \tag{3.27}$$

where m_1 depends on the quadrature rule and $\hat{\eta}$ is inversely proportional to n_i. For example, we may consider n_i-panel composite trapezoidal rule on $\hat{\omega}_i$. It is well known that for n_i-panel composite trapezoidal rule, (3.27) is true with $m_1 = 2$ and $\hat{\eta} = n_i^{-2}/12$. We may also consider an n_i-point Gaussian quadrature rule, in which case we have $\hat{\eta} = O(n_i^{-m_1})$. Now from (3.26) and (3.27), we have

$$\left|\int_{\hat{\omega}_i} \hat{\varrho}(\xi)\,d\xi - ⨍_{\hat{\omega}_i} \hat{\varrho}(\xi)\,d\xi\right| \leqslant \eta\|\hat{\varrho}\|_{L_\infty(\hat{\omega}_i)}, \quad |\alpha| = m_1, \tag{3.28}$$

where $\eta = C\hat{\eta}$ is inversely proportional to n_i. Thus we can choose a quadrature rule (i.e., number of quadrature points n_i) such that the associated η is small. Finally, we get (3.10) by employing a standard scaling argument to the inequality (3.28). Using similar arguments, we can show that we can choose a quadrature rule with a small τ in (3.13). We further note that the functions ϱ and ϑ (in (3.10) and (3.11) respectively) that we numerically integrate in this paper satisfy the conditions (3.12) and (3.13).

Remark 3.4. For each $i \in N_h$, we define the linear functional $G_i : H^2(\omega_i) \to \mathbb{R}$ as follows:

$$G_i(v) = \int_{\omega_i} \nabla v \cdot \nabla\phi_i\,dx + \int_{\omega_i} \Delta v\,\phi_i\,dx - \int_{\partial\omega_i\cap\Gamma} \nabla v \cdot \vec{n}\,\phi_i\,ds. \tag{3.29}$$

It follows directly from Green's formula that

$$G_i(p) = 0, \quad \forall\, p \in \mathscr{P}^k. \tag{3.30}$$

In fact, (3.30) is true for any smooth function p. The linear functional G_i^*, defined in (3.17), is obtained by using numerical integration on each integral in G_i. In general, (3.30) is not true if G_i is replaced by G_i^*. In (3.18) of assumption **QA3**, we require the exact same property to hold for G_i^*.

Remark 3.5. It is instructive to illustrate the assumption **QA3**, i.e., (3.18) in simpler situations. Let $\Omega \subset \mathbb{R}^2$ and $k = 1$. Considering $p(x_1,x_2) = x_1$ in (3.18), we get

$$G_i^*(x_1) = ⨍_{\omega_i} \frac{\partial\phi_i}{\partial x_1}\,dx - ⨍_{\partial\omega_i\cap\Gamma} n_1\phi_i\,ds = 0, \quad i \in N_h, \tag{3.31}$$

where $\vec{n} = (n_1,n_2)$. Similarly, considering $p(x_1,x_2) = x_2$ in (3.18), we get

$$G_i^*(x_2) = ⨍_{\omega_i} \frac{\partial\phi_i}{\partial x_2}\,dx - ⨍_{\partial\omega_i\cap\Gamma} n_2\phi_i\,ds = 0, \quad i \in N_h. \tag{3.32}$$

Thus for $k = 1$, the quadrature must satisfy the two conditions (3.31) and (3.32) for each $i \in N_h$. In particular, the quadrature must satisfy

$$⨍_{\omega_i} \nabla\phi_i\,dx = 0, \quad \forall i \in N''_h. \tag{3.33}$$

We illustrate now (3.18) for $k = 2$. Considering $p(x_1,x_2) = x_1^2$ in (3.18), we get

$$G_i^*(x_1^2) = 2\left[⨍_{\omega_i} x_1\frac{\partial\phi_i}{\partial x_1}\,dx + ⨍_{\omega_i}\phi_i\,dx - ⨍_{\partial\omega_i\cap\Gamma} x_1 n_1\,\phi_i\,ds\right] = 0, \quad i \in N_h. \tag{3.34}$$

Similarly, considering $p(x_1,x_2) = x_1x_2$ and $p(x_1,x_2) = x_2^2$ in (3.18), we get

$$G_i^*(x_1x_2) = ⨍_{\omega_i}\left(x_2\frac{\partial\phi_i}{\partial x_1} + x_1\frac{\partial\phi_i}{\partial x_2}\right)dx - ⨍_{\partial\omega_i\cap\Gamma}(x_2 n_1 + x_1 n_2)\,\phi_i\,ds = 0, \tag{3.35}$$

and

$$G_i^*(x_2^2) = 2\left[⨍_{\omega_i} x_2\frac{\partial\phi_i}{\partial x_2}\,dx + ⨍_{\omega_i}\phi_i\,dx - ⨍_{\partial\omega_i\cap\Gamma} x_2 n_2\,\phi_i\,ds\right] = 0, \tag{3.36}$$

for $i \in N_h$. Thus for $k = 2$, the quadrature must satisfy (3.34)–(3.36) in addition to the assumptions (3.31) and (3.32). We will present quadrature schemes satisfying assumption **QA3** later in this paper.

4. Effect of numerical integration

In this section, we will study the effect of quadrature on the meshless method. In particular, we will compare $u - u_h^*$ with $u - u_h$, where u, u_h, and u_h^* are defined in problems (2.5), (2.23) and (3.8) respectively. We will assume u to be smooth; in particular, $u \in C^{k+1}(\overline{\Omega})$. This assumption will allow us to present the main ideas simply and effectively. We will first prove the so called Strang lemma.

Lemma 4.1. *Suppose* (u_h,λ_h) *and* (u_h^*,λ_h^*) *are the solutions of problems* (2.23) and (3.8) *respectively. Let* $(w,\tilde{\lambda}) \in V_\Psi^h$ *be arbitrary. Then there exists* C*, independent of* h*, such that*

$$\|(u_h - u_h^*, \lambda_h - \lambda_h^*)\|_{V_\Psi^h} \leqslant C\left[\|(u_h - w, \lambda_h - \tilde{\lambda})\|_{V_\Psi^h} + \sup_{(v,\mu)\in V_\Psi^h} \frac{|[B(w,v) - B^*(w,v)] + [L^*(v) - L(v)]|}{\|(v,\mu)\|_{V_\Psi^h}}\right]. \tag{4.1}$$

Proof. We first note that $(u_h^* - w, \lambda_h^* - \tilde{\lambda}) \in V_\Psi^h$. Therefore, from the inf–sup condition (3.15) we get

$$\|(u_h^* - w, \lambda_h^* - \tilde{\lambda})\|_{V_\Psi^h} \leqslant C \sup_{(v,\mu)\in V_\Psi^h} \frac{\mathscr{B}^*_\Psi(u_h^* - w, \lambda_h^* - \tilde{\lambda}; v,\mu)}{\|(v,\mu)\|_{V_\Psi^h}}. \tag{4.2}$$

Now,

$$\begin{aligned}&\mathscr{B}^*_\Psi(u^*_h - w, \lambda^*_h - \tilde{\lambda}; v, \mu)\\&\quad = \mathscr{B}_\Psi(u_h - w, \lambda_h - \tilde{\lambda}; v, \mu) + \mathscr{B}_\Psi(w, \tilde{\lambda}; v, \mu) - \mathscr{B}^*_\Psi(w, \tilde{\lambda}; v, \mu)\\&\qquad + \mathscr{B}^*_\Psi(u^*_h, \lambda^*_h; v, \mu) - \mathscr{B}_\Psi(u_h, \lambda_h; v, \mu)\\&\quad = \mathscr{B}_\Psi(u_h - w, \lambda_h - \tilde{\lambda}; v, \mu) + B(w, v) - B^*(w, v) + L^*(v) - L(v).\end{aligned}$$

Therefore, from (4.2) and (3.14), we get

$$\begin{aligned}\|(u^*_h - w, \lambda^*_h - \tilde{\lambda})\|_{V^h_\Psi} &\leqslant C \sup_{(v,\mu)\in V^h_\Psi} \frac{1}{\|(v,\mu)\|_{V^h_\Psi}} |\mathscr{B}_\Psi(u_h - w, \lambda_h - \tilde{\lambda}; v, \mu)\\&\qquad + B(w, v) - B^*(w, v) + L^*(v) - L(v)|\\&\leqslant C\|(u_h - w, \lambda_h - \tilde{\lambda})\|_{V^h_\Psi}\\&\qquad + C \sup_{(v,\mu)\in V^h_\Psi} \frac{|B(w, v) - B^*(w, v) + L^*(v) - L(v)|}{\|(v,\mu)\|_{V^h_\Psi}}.\end{aligned}$$

Finally, using the triangle inequality and the above, we get

$$\begin{aligned}&\|(u_h - u^*_h, \lambda_h - \lambda^*_h)\|_{V^h_\Psi}\\&\quad \leqslant \|(u_h - w, \lambda_h - \tilde{\lambda})\|_{V^h_\Psi} + \|(u^*_h - w, \lambda^*_h - \tilde{\lambda})\|_{V^h_\Psi}\\&\quad \leqslant C\Big[\|(u_h - w, \lambda_h - \tilde{\lambda})\|_{V^h_\Psi}\\&\qquad + \sup_{(v,\mu)\in V^h_\Psi} \frac{|[B(w, v) - B^*(w, v)] + [L^*(v) - L(v)]|}{\|(v,\mu)\|_{V^h_\Psi}}\Big],\end{aligned}$$

which is the desired result. □

In the analysis presented of this section, we will apply (4.1) with $w = \mathscr{I}_h u$ and estimate each term on the right-hand side of (4.1). Recall that $\mathscr{I}_h u$ is the V_h-interpolant of u, as defined in (2.24). From the interpolation error estimate (2.25), we have

$$\begin{aligned}\|\mathscr{I}_h u\|_{W^{k+1,\infty}(\Omega)} &\leqslant \|u\|_{W^{k+1,\infty}(\Omega)} + \|u - \mathscr{I}_h u\|_{W^{k+1,\infty}(\Omega)}\\&\leqslant C\|u\|_{W^{k+1,\infty}(\Omega)}.\end{aligned} \tag{4.3}$$

For a smooth function v and $i \in N_h$, let

$$T^k_i v \equiv \sum_{|\alpha|\leqslant k} \frac{D^\alpha v(\bar{x}_i)}{\alpha!} (x - \bar{x}_i)^\alpha \tag{4.4}$$

be the kth degree Taylor polynomial of v centered at $\bar{x}_i$, where $\bar{x}_i$ is the center of the ball $o_i \subset \omega_i$ (recall that ω_i is star-shaped with respect to o_i). It is well known that [7]

$$|v - T^k_i v|_{W^{j,\infty}(\omega_i)} \leqslant \frac{Ch^{k+1-j}}{(k+1-j)!} \|v\|_{W^{k+1,\infty}(\omega_i)}, \quad j = 0, 1, \ldots, k+1. \tag{4.5}$$

Now consider $T^k_i \mathscr{I}_h u$ – the kth degree Taylor polynomial of $\mathscr{I}_h u$ centered at $\bar{x}_i$. We set

$$R_i \equiv \mathscr{I}_h u - T^k_i \mathscr{I}_h u, \quad i \in N_h. \tag{4.6}$$

Then from (4.3) and (4.5) with $v = \mathscr{I}_h u$, we get

$$|R_i|_{W^{j,\infty}(\omega_i)} \leqslant Ch^{k+1-j}\|\mathscr{I}_h u\|_{W^{k+1,\infty}(\omega_i)} \leqslant Ch^{k+1-j}\|u\|_{W^{k+1,\infty}(\Omega)}. \tag{4.7}$$

We will use this estimate later for $j = 1, 2$ in the next lemma.

Lemma 4.2. *For $i \in N_h$, let G_i and G^*_i be the linear functionals as defined (3.29) and (3.17) respectively. Then there exists a positive constant C, independent of i and h, such that*

$$|G_i(\mathscr{I}_h u) - G^*_i(\mathscr{I}_h u)| \leqslant \begin{cases} C(\eta+\tau)h^{k+d-1}\|u\|_{W^{k+1,\infty}(\Omega)} & i \in N'_h;\\ C\eta h^{k+d-1}\|u\|_{W^{k+1,\infty}(\Omega)} & i \in N''_h.\end{cases}$$

Proof. For each $i \in N_h$, we write $\mathscr{I}_h u$ as

$$\mathscr{I}_h u = T^k_i \mathscr{I}_h u + R_i,$$

where R_i is the remainder defined in (4.6). Since $T^k_i \mathscr{I}_h u$ is a polynomial of degree k, we have $G_i(T^k_i \mathscr{I}_h u) = G^*_i(T^k_i \mathscr{I}_h u) = 0$ from (3.30) and (3.18) and therefore,

$$\begin{aligned}G_i(\mathscr{I}_h u) - G^*_i(\mathscr{I}_h u) &= G_i(T^k_i \mathscr{I}_h u + R_i) - G^*_i(T^k_i \mathscr{I}_h u + R_i)\\&= G_i(R_i) - G^*_i(R_i).\end{aligned} \tag{4.8}$$

Let $i \in N'_h$. Then, using (3.10) and (3.11), and the assumption **A2**,

$$\begin{aligned}|G_i(R_i) - G^*_i(R_i)| &\leqslant \left|\int_{\omega_i} \nabla R_i \cdot \nabla\phi_i\, dx - \fint_{\omega_i} \nabla R_i \cdot \nabla\phi_i\, dx\right|\\&\quad + \left|\int_{\omega_i} \Delta R_i\, \phi_i\, dx - \fint_{\omega_i} \Delta R_i\, \phi_i\, dx\right|\\&\quad + \left|\int_{\partial\omega_i\cap\Gamma} \nabla R_i \cdot \vec{n}\, \phi_i\, ds - \fint_{\partial\omega_i\cap\Gamma} \nabla R_i \cdot \vec{n}\, \phi_i\, ds\right|\\&\leqslant \eta|\omega_i|\,\|\nabla R_i \cdot \nabla\phi_i\|_{L_\infty(\omega_i)} + \eta|\omega_i|\,\|\Delta R_i\, \phi_i\|_{L_\infty(\omega_i)}\\&\quad + \tau|\partial\omega_i \cap \Gamma|\,\|\nabla R_i \cdot \vec{n}\, \phi_i\|_{L_\infty(\partial\omega_i\cap\Gamma)}\\&\leqslant C\eta h^{d-1}|R_i|_{W^{1,\infty}(\omega_i)} + C\eta h^d |R_i|_{W^{2,\infty}(\omega_i)}\\&\quad + C\tau h^{d-1}|R_i|_{W^{1,\infty}(\omega_i)} \leqslant C(\eta+\tau)h^{k+d-1}\|u\|_{W^{k+1,\infty}(\Omega)},\end{aligned} \tag{4.9}$$

where we used (4.7) to obtain the last inequality.

For $i \in N''_h$, we have $\overline{\omega}_i \subset \Omega$ and therefore $\phi_i|_{\partial\omega_i} = 0$. Now following the arguments leading to (4.9), we get

$$|G_i(R_i) - G^*_i(R_i)| \leqslant C\eta h^{k+d-1}\|u\|_{W^{k+1,\infty}(\Omega)}, \quad i \in N''_h.$$

Thus from (4.8) and (4.9), we get the desired result. □

We now prove the main result of this paper.

Theorem 4.1. *Suppose the approximating subspace V_h and the numerical integration scheme satisfy conditions* **A1–A5** *and* **QA1–QA3**, *respectively. Then for small η, there is a positive constant C, independent of u, η, τ, and h, such that*

$$|u - u^*_h|_{H^1(\Omega)} \leqslant C[h^k + (\eta+\tau)h^k + \eta h^{k-1}]\,\|u\|_{W^{k+1,\infty}(\Omega)}.$$

Proof. Let $\mathscr{I}_h u$ be the V_h-interpolant of u. We note that $B(D, v) = 0$ and recall that $B^*(D, v) = 0$ for an arbitrary constant D (see (3.6)). We then substitute $(w, \tilde{\lambda}) = (\mathscr{I}_h u + D, \lambda_h)$ in Lemma 4.1 to get

$$\begin{aligned}&\|(u_h - u^*_h, \lambda_h - \lambda^*_h)\|_{V^h_\Psi}\\&\quad \leqslant C\Big[\|u_h - \mathscr{I}_h u - D\|_{H^1(\Omega)}\\&\qquad + \sup_{(v,\mu)\in V^h_\Psi} \frac{|[B(\mathscr{I}_h u, v) - B^*(\mathscr{I}_h u, v)] + [L^*(v) - L(v)]|}{\|(v,\mu)\|_{V^h_\Psi}}\Big].\end{aligned} \tag{4.10}$$

We will now estimate the right-hand side of (4.10).

Since the solution u is smooth, we have for $i \in N_h$,

$$\begin{aligned}\int_{\omega_i} f\,\phi_i\, dx &= -\int_{\omega_i} \Delta u\, \phi_i\, dx\\&= -\int_{\omega_i} \Delta(\mathscr{I}_h u)\,\phi_i\, dx + \int_{\omega_i} \Delta(\mathscr{I}_h u - u)\,\phi_i\, dx\end{aligned}$$

and

$$\begin{aligned}\int_{\partial\omega_i\cap\Gamma} g\phi_i\, ds &= \int_{\partial\omega_i\cap\Gamma} \nabla u \cdot \vec{n}\, \phi_i\, ds\\&= \int_{\partial\omega_i\cap\Gamma} \nabla(\mathscr{I}_h u) \cdot \vec{n}\, \phi_i\, ds + \int_{\partial\omega_i\cap\Gamma} \nabla(u - \mathscr{I}_h u) \cdot \vec{n}\, \phi_i\, ds.\end{aligned}$$

Therefore, using the definition of the linear functional G_i (see (3.29)) we get,

$$
\begin{aligned}
B(\mathscr{I}_h u, \phi_i) - L(\phi_i) &= B(\mathscr{I}_h u, \phi_i) - \int_{\omega_i} f\,\phi_i\,dx - \int_{\partial\omega_i\cap\Gamma} g\,\phi_i\,ds \\
&= \int_{\omega_i} \nabla(\mathscr{I}_h u)\cdot\nabla\phi_i\,dx + \int_{\omega_i} \Delta(\mathscr{I}_h u)\,\phi_i\,dx \\
&\quad - \int_{\partial\omega_i\cap\Gamma} \nabla(\mathscr{I}_h u)\cdot\vec{n}\,\phi_i\,ds + \int_{\omega_i}\Delta(u-\mathscr{I}_h u)\,\phi_i\,dx \\
&\quad - \int_{\partial\omega_i\cap\Gamma}\nabla(u-\mathscr{I}_h u)\cdot\vec{n}\,\phi_i\,ds = G_i(\mathscr{I}_h u) \\
&\quad + \int_{\omega_i}\Delta e_I\,\phi_i\,dx - \int_{\partial\omega_i\cap\Gamma}\nabla e_I\cdot\vec{n}\,\phi_i\,ds, \qquad (4.11)
\end{aligned}
$$

where $e_I \equiv u - \mathscr{I}_h u$. Likewise, repeating the argument leading to (4.11) with $\int$ replaced by $⨍$, we get for $i \in N_h$,

$$
B^*(\mathscr{I}_h u, \phi_i) - L^*(\phi_i) = G_i^*(\mathscr{I}_h u) + ⨍_{\omega_i}\Delta e_I\,\phi_i\,dx - ⨍_{\partial\omega_i\cap\Gamma}\nabla e_I\cdot\vec{n}\,\phi_i\,ds, \qquad (4.12)
$$

where G_i^* is the linear functional defined in (3.17). Therefore combining (4.11) and (4.12), we get for $i \in N_h$,

$$
\begin{aligned}
&B(\mathscr{I}_h u, \phi_i) - B^*(\mathscr{I}_h u, \phi_i) + L^*(\phi_i) - L(\phi_i) \\
&\quad = G_i(\mathscr{I}_h u) - G^*(\mathscr{I}_h u) + \int_{\omega_i}\Delta e_I\,\phi_i\,dx - ⨍_{\omega_i}\Delta e_I\,\phi_i\,dx \\
&\qquad - \int_{\partial\omega_i\cap\Gamma}\nabla e_I\cdot\vec{n}\,\phi_i\,ds + ⨍_{\partial\omega_i\cap\Gamma}\nabla e_I\cdot\vec{n}\,\phi_i\,ds. \qquad (4.13)
\end{aligned}
$$

Let $i \in N_h'$. Then using (3.10) and (3.11), Lemma 4.2, (2.25), and assumption **A2** in (4.13), we have

$$
\begin{aligned}
&|B(\mathscr{I}_h u, \phi_i) - B^*(\mathscr{I}_h u, \phi_i) + L^*(\phi_i) - L(\phi_i)| \\
&\quad \leqslant |G_i(\mathscr{I}_h u) - G_i^*(\mathscr{I}_h u)| + \eta|\omega_i|\,\|\Delta e_I\,\phi_i\|_{L_\infty(\omega_i)} \\
&\qquad + \tau|\partial\omega_i\cap\Gamma|\,\|\nabla e_I\cdot\vec{n}\,\phi_i\|_{L_\infty(\partial\omega_i)} \\
&\quad \leqslant C(\eta+\tau)h^{k+d-1}\|u\|_{W^{k+1,\infty}(\Omega)}, \quad \forall\ i \in N_h'. \qquad (4.14)
\end{aligned}
$$

Now let $i \in N_h''$, so $\phi_i|_{\partial\omega_i} = 0$ and using (3.10), Lemma 4.2, (2.25), and assumption **A2** in (4.13), we have

$$
\begin{aligned}
&|B(\mathscr{I}_h u, \phi_i) - B^*(\mathscr{I}_h u, \phi_i) + L^*(\phi_i) - L(\phi_i)| \\
&\quad \leqslant |G_i(\mathscr{I}_h u) - G_i^*(\mathscr{I}_h u)| + \eta|\omega_i|\,\|\Delta e_I\,\phi_i\|_{L_\infty(\omega_i)} \\
&\quad \leqslant C\eta h^{k+d-1}\|u\|_{W^{k+1,\infty}(\Omega)}, \quad \forall\ i \in N_h''. \qquad (4.15)
\end{aligned}
$$

We now estimate the second term of the RHS of (4.10). Let $v = \sum_{i\in N_h} v_i\phi_i$ be an arbitrary element in V_h. Then from (4.14), (4.15), (2.16), (2.17) and (2.20), and a trace-inequality, we have

$$
\begin{aligned}
&|B(\mathscr{I}_h u, v) - B^*(\mathscr{I}_h u, v) + L^*(v) - L(v)| \\
&\quad \leqslant \left|\sum_{i\in N_h''} v_i[B(\mathscr{I}_h u,\phi_i) - B^*(\mathscr{I}_h u,\phi_i) + L^*(\phi_i) - L(\phi_i)]\right| \\
&\qquad + \left|\sum_{i\in N_h'} v_i[B(\mathscr{I}_h u,\phi_i) - B^*(\mathscr{I}_h u,\phi_i) + L^*(\phi_i) - L(\phi_i)]\right| \\
&\quad \leqslant \left(\sum_{i\in N_h''} v_i^2\right)^{1/2}\left(\sum_{i\in N_h''}|B(\mathscr{I}_h u,\phi_i) - B^*(\mathscr{I}_h u,\phi_i) + L^*(\phi_i) - L(\phi_i)|^2\right)^{1/2} \\
&\qquad + \left(\sum_{i\in N_h'} v_i^2\right)^{1/2}\left(\sum_{i\in N_h'}|B(\mathscr{I}_h u,\phi_i) - B^*(\mathscr{I}_h u,\phi_i) + L^*(\phi_i) - L(\phi_i)|^2\right)^{1/2} \\
&\quad \leqslant C\eta h^{k+d-1}|N_h''|^{1/2}\left(\sum_{i\in N_h''} v_i^2\right)^{1/2}\|u\|_{W^{k+1,\infty}(\Omega)} \\
&\qquad + C(\eta+\tau)h^{k+d-1}|N_h'|^{1/2}\left(\sum_{i\in N_h'} v_i^2\right)^{1/2}\|u\|_{W^{k+1,\infty}(\Omega)} \leqslant Ch^{k-1}[\eta\|v\|_{L_2(\Omega)} \\
&\qquad + h(\eta+\tau)\|v\|_{L_2(\Gamma)}]\|u\|_{W^{k+1,\infty}(\Omega)} \\
&\quad \leqslant Ch^{k-1}[\eta + h(\eta+\tau)]\|u\|_{W^{k+1,\infty}(\Omega)}\|v\|_{H^1(\Omega)}. \qquad (4.16)
\end{aligned}
$$

Then, from (4.10) and the Poincaré inequality, we get

$$
\begin{aligned}
|u_h - u_h^*|_{H^1(\Omega)} &\leqslant \|(u_h - u_h^*, \lambda_h - \lambda_h^*)\|_{V_\Psi^h} \\
&\leqslant C\inf_{D\in\mathbb{R}}\|u_h - \mathscr{I}_h u - D\|_{H^1(\Omega)} \\
&\quad + Ch^{k-1}[\eta + h(\eta+\tau)]\|u\|_{W^{k+1,\infty}(\Omega)}\sup_{(v,\mu)\in V_\Psi^h}\frac{\|v\|_{H^1(\Omega)}}{\|(v,\mu)\|_{V_\Psi^h}} \\
&\leqslant C|u_h - \mathscr{I}_h u|_{H^1} + C[(\eta+\tau)h^k + \eta h^{k-1}]\|u\|_{W^{k+1,\infty}(\Omega)}. \qquad (4.17)
\end{aligned}
$$

Finally, from (2.27) and (2.28)

$$
\begin{aligned}
|u - u_h^*|_{H^1(\Omega)} &\leqslant |u - u_h|_{H^1(\Omega)} + |u_h - u_h^*|_{H^1(\Omega)} \\
&\leqslant C[h^k + (\eta+\tau)h^k + \eta h^{k-1}]\|u\|_{W^{k+1,\infty}(\Omega)},
\end{aligned}
$$

which is the desired result. □

For $k = 1$, we only require the quadrature to satisfy a reduced form of **QA3**, namely, we assume that (3.18) of **QA3** is satisfied only for $i \in N_h''$.

Theorem 4.2. *Suppose the approximating subspace V_h satisfies conditions* **A1**–**A5** *with $k = 1$. We consider numerical integration scheme satisfying* **QA1**–**QA3**, *but (3.18) of* **QA3** *is satisfied only for $i \in N_h''$. Then for small η, there is a constant C, independent of u, η, τ, and h, such that*

$$
|u - u_h^*|_{H^1(\Omega)} \leqslant C[h + \eta + \tau]\,\|u\|_{W^{2,\infty}(\Omega)}.
$$

The proof of this result can be obtained by slightly modifying the proof of Theorem 4.1; we do not provide the details here.

Remark 4.1. It is clear from Theorem 4.1 that we do not have optimal order of convergence, i.e., $|u - u_h^*|_{H^1(\Omega)} = O(h^{k-1}[h+\eta])$. But if we consider $\eta \leqslant Ch$, then we get

$$
|u - u_h^*|_{H^1(\Omega)} = O(h^k).
$$

This means that if we increase the accuracy of the quadrature as h becomes smaller, we restore the optimal order of convergence. This effect of numerical integration in meshless method is very different from the effect of numerical integration in FEM.

5. Numerical results

In this section, we present computational data illuminating the results in Section 4 in one dimension. We will also develop numerical integration rules satisfying (3.18) of assumption **QA3**.

We consider the one dimensional version of the problem (2.5) with $\Omega = (0,1)$. Let $u(x) = e^x - (e-1)$, satisfying $\Phi(u) = \int_0^1 u\,dx = 0$, be the exact solution of (2.5) with $L(v) = -\int_0^1 e^x v(x)\,dx + e\,v(1) - v(0)$. To approximate this solution by the meshless method (2.23), we first construct a $C^2(\mathbb{R})$, symmetric, RKP *basic shape function* $\phi(x)$ with support $[-R, R]$, satisfying

$$
\sum_{j\in\mathbb{Z}}\phi(x-j) = 1 \quad\text{and}\quad \sum_{j\in\mathbb{Z}} j\,\phi(x-j) = x, \quad \forall x\in\mathbb{R},
$$

with $R = 1.8$ (see [2,18,19]). We then consider a natural number $N > 1$, and for $h = 1/N$, we let

$$
\bar{N}_h = \{x_i = ih:\ i = -1, 0, 1, \ldots, N, N+1\}.
$$

For each $x_i \in \bar{N}_h$, we define the shape function

$$
\phi_i(x) \equiv \phi\left(\frac{x}{h} - i\right), \quad x \in \Omega \equiv (0,1). \qquad (5.1)
$$

Then for $i \in N_h, \omega_i \equiv (\alpha_i, \beta_i) = (ih - Rh, ih + Rh)\cap\Omega$ and supp $\phi_i(x) = \overline{\omega}_i = [\alpha_i, \beta_i]\cap\Omega$. We note that for $i = 2, 3, \ldots, N-2$, we have

$$(ih - Rh, ih + Rh) \subset \Omega, \quad \alpha_i = ih - Rh,$$
$$\beta_i = ih + Rh \quad \text{and} \quad \phi_i(\alpha_i) = \phi_i(\beta_i) = 0. \tag{5.2}$$

Thus $N''_h = \{2, 3, \ldots, N-2\}$ and $N'_h = \{-1, 0, 1, N-1, N, N+1\}$.

It can be easily shown that the shape functions $\{\phi_i\}_{i=-1}^{N+1}$ *reproduce polynomials of degree* $k = 1$, i.e.,

$$\sum_{i=-1}^{N+1} p(x_i)\,\phi_i(x) = p(x), \quad \forall p \in \mathscr{P}^1(\Omega).$$

We next show a procedure to obtain a quadrature scheme that satisfies the condition (3.18) of the assumption **QA3**. Suppose $f(x)$ is smooth in $[\alpha_i, \beta_i]$ and let $I_i(f) \equiv \int_{\alpha_i}^{\beta_i} f(x)\,dx$. To approximate $I_i(f)$, we seek a p-point quadrature rule of the form

$$Q^i_{gc}(f) \equiv \sum_{s=1}^{p} \bar{w}_s f(\bar{z}_s) \quad (\bar{z}_s \in [\alpha_i, \beta_i] \quad \text{and} \quad \bar{w}_s \text{ depend on } i) \tag{5.3}$$

with the property that

$$Q^i_{gc}(\phi'_i) = 0, \quad i \in N''_h. \tag{5.4}$$

This is precisely the condition (3.18) in $1-d$ for $k = 1$ (see (3.33) in Remark 3.5). We start with a p-point quadrature rule for the interval $[\alpha_i, \beta_i]$ of the form

$$Q^i_g(f) \equiv \sum_{s=1}^{p} w_s f(z_s). \tag{5.5}$$

We now define $\bar{z}_s \equiv z_s$ and

$$\bar{w}_s \equiv w_s + \theta_i\, w_s\, \phi'_i(z_s) \tag{5.6}$$

in (5.3), and choose θ_i such that (5.4) is satisfied. We first note that

$$Q^i_{gc}(\phi'_i) = \sum_{s=1}^{p} \bar{w}_s \phi'_i(\bar{z}_s) = \sum_{s=1}^{p} \left[w_s + \theta_i\, w_s\, \phi'_i(z_s)\right] \phi'_i(z_s)$$
$$= \sum_{s=1}^{p} w_s \phi'_i(z_s) + \theta_i \sum_{s=1}^{p} w_s\, [\phi'_i(z_s)]^2.$$

Thus imposing condition (5.4), we get

$$\sum_{s=1}^{p} w_s \phi'_i(z_s) + \theta_i \sum_{s=1}^{p} w_s\, [\phi'_i(z_s)]^2 = 0 \quad \text{or,}\ \theta_i = \frac{-\sum_{s=1}^{p} w_s \phi'_i(z_s)}{\sum_{s=1}^{p} w_s\, [\phi'_i(z_s)]^2}. \tag{5.7}$$

Thus $Q^i_{gc}(f)$ satisfies the condition (5.4) and we refer to $Q^i_{gc}(f)$ as the p-point ***corrected quadrature***.

We now consider the quadrature rule $Q^i_g(f)$ in (5.5) to be the p-point Gauss quadrature rule. It is well known that the points $\{z_s\}_{s=1}^{p}$ are symmetrically placed in the interval (α_i, β_i) about the mid-point $m_i \equiv (\alpha_i + \beta_i)/2$; the weights $\{w_s\}_{s=1}^{p}$ are also "symmetric", i.e., $w_s = w_{p+1-s}$, $s = 1, 2, \ldots, p$. We next recall that the shape functions $\phi_i(x)$, defined in (5.1) are symmetric in the interval (α_i, β_i) about the mid-point m_i. Consequently, $\phi'_i(x)$ is anti-symmetric in the interval (α_i, β_i) about m_i. Therefore, it is clear that

$$Q^i_g(\phi'_i) = \sum_{s=1}^{p} w_s \phi'_i(z_s) = 0, \quad \forall i \in N''_h.$$

Thus the Gaussian quadrature $Q^i_g(f)$ satisfies (5.4); in fact, $\theta_i = 0$ in this situation and $Q^i_g(f) = Q^i_{gc}(f)$.

We now present numerical experiments to illuminate the results in Theorem 4.1 for $k = 1$; in particular we illuminate the result in Theorem 4.2. We considered $u = e^x - (e-1)$ to be exact solution of (2.5) with $\Phi(v) \equiv \int_0^1 v\,dx$. The function u was approximated by $u^*_h \in V_h = \mathrm{span}\{\phi_i(x)\}_{i=-1}^{N+1}$ – the solution of the meshless method with numerical integration (3.8), where ϕ_i is defined in (5.1). We recall that the linear functional $\Psi(\cdot)$ was used in the meshless method (3.8) to compute u^*_h. We employed the p-point Gauss quadrature rule $Q^i_g(f)$ to numerically integrate the relevant terms, e.g., γ^*_{ij} and l^*_i (see (3.2) and (3.3)). We note that we did not approximate the boundary term in (3.3) in our $1-d$ example and so have $\tau = 0$.

Table 1
Standard p-point Gauss rule.

h	$\lvert u - u^*_h\rvert_{H^1(\Omega)}$		
	$p = 8$	$p = 16$	$p = 32$
1/10	1.3480E−02	3.4015E−03	3.3906E−03
1/20	1.2856E−02	1.7655E−03	1.7426E−03
1/40	1.2527E−02	9.3409E−04	8.8369E−04
1/80	1.2364E−02	5.4747E−04	4.4516E−04
1/160	1.2284E−02	3.9752E−04	2.2374E−04
1/320	1.2244E−02	3.5257E−04	1.1285E−04
1/640	1.2224E−02	3.4182E−04	5.7922E−05
1/1280	1.2214E−02	3.3982E−04	3.1563E−05

The H^1-seminorm of the error, $\lvert u - u^*_h\rvert_{H^1(\Omega)}$, where $u(x) = e^x - (e-1)$ and u^*_h is the approximate solution obtained using *standard Gaussian quadrature*. The shape functions reproduce polynomial of degree $k = 1$.

We used $p = 8, 16$, and 32 in $Q^i_g(f)$ and computed the seminorm $\lvert u - u^*_h\rvert_{H^1(\Omega)}$. We note that η decreases as p increases. We presented these results in Table 1. We also present the log–log graph of $\lvert u - u^*_h\rvert_{H^1(\Omega)}$ with respect to h in Fig. 1.

We observe from Table 1 that the error $\lvert u - u^*_h\rvert_{H^1(\Omega)}$ decreases as h decreases. Moreover, for $p = 16$, we observe from Fig. 1 that $\lvert u - u^*_h\rvert_{H^1(\Omega)} = O(h)$ at the beginning, but "levels off" for smaller values of h. For $p = 32$ the pattern is same, but the error is $O(h)$ for few more smaller values of h. This pattern suggests that $\lvert u - u^*_h\rvert_{H^1(\Omega)} = O(h + \eta)$.

We will now show that the error $\lvert u - u^*_h\rvert_{H^1(\Omega)}$ is not $O(h + \eta)$ when the underlying quadrature rule does not satisfy the assumption (5.4); we will show that the error increases as h becomes smaller. We construct a quadrature rule on (α_i, β_i) such that the quadrature points are not situated symmetrically about the mid-point m_i.

Consider the mapping $h : [-1, 1] \to [-1, 1]$ given by

$$y = h(z) = z + 0.1(z^2 - 1).$$

Clearly,

$$h'(z) = 1 + 0.2z > 0 \quad \forall z \in [-1, 1].$$

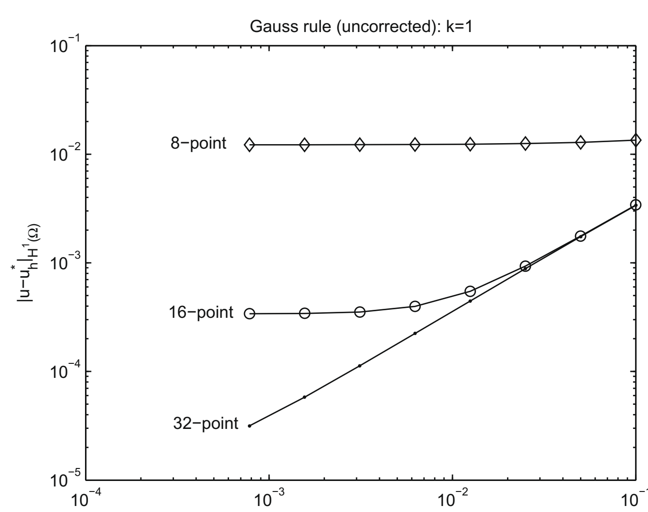

Fig. 1. The loglog plot of $\lvert u - u^*_h\rvert_{H^1(\Omega)}$ with respect to h.u^*_h is the approximate solution obtained using p-point *standard Gaussian quadrature (symmetric)* with $p = 8, 16$, and 32.

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Consider

$$I(g) \equiv \int_{-1}^{1} g(y)\,dy = \int_{-1}^{1} g(h(z))h'(z)\,dz$$

The integral on the right could be approximated by the standard p-point Gauss quadrature, given by

$$I(g) \approx \sum_{s=1}^{p} \upsilon_s g(h(\zeta_s))h'(\zeta_s),$$

where $\{\zeta_s\}$ and $\{\upsilon_s\}$ are standard Gauss points and Gauss weights respectively for the interval $(-1,1)$. This induces an associated p-point non-symmetric Gauss quadrature $Q_{ns}(g)$ on $(-1,1)$ to approximate $I(g)$ given by

$$Q_{ns}(g) \equiv \sum_{s=1}^{p} \upsilon_s^{ns} g(\zeta_s^{ns}),$$

where

$$\upsilon_s^{ns} \equiv \upsilon_s h'(\zeta_s), \quad \zeta_s^{ns} \equiv h(\zeta_s).$$

Clearly υ_s^{ns} and ζ_s^{ns} are not symmetric. It is well known that the "precision" of standard p-point Gauss quadrature is $(2p-1)$. It can be easily shown that the "precision" of the associated p-point non-symmetric Gauss quadrature is $(p-1)$.

The p-point quadrature $Q_{ns}(\cdot)$ induces the associated p-point ***non-symmetric Gauss quadrature*** $Q^i_{ns}(\cdot)$ for the interval (α_i, β_i) given by

$$Q^i_{ns}(f) \equiv \sum_{s=1}^{p} w_s^{ns} f(z_s^{ns}),$$

where

$$z_s^{ns} = \frac{\beta_i - \alpha_i}{2}\zeta_s^{ns} + \frac{\beta_i + \alpha_i}{2} \quad \text{and} \quad w_s^{ns} = \frac{\beta_i - \alpha_i}{2}\upsilon_s^{ns}. \tag{5.8}$$

We note that

$$Q^i_{ns}(\phi'_i) \neq 0, \quad \text{for } i \in N''_h.$$

Thus $Q^i_{ns}(f)$ does not satisfy the assumption (5.4).

We computed u_h^* – the solution of the meshless method (3.8) with numerical integration, where we used p-point $Q^i_{ns}(f)$ to compute the relevant integrals. We computed the error $|u - u_h^*|_{H^1(\Omega)}$ for $p = 8, 16, 32$, and 64, and presented the data in Table 2. We also present the log-log graph of $|u - u_h^*|_{H^1(\Omega)}$ with respect to h in Fig. 2.

We observe from Table 2 and Fig. 2 that for $p = 8$, the error increases as h decreases and it "levels off" for smaller values of h. For $p = 16, 32$, and 64, the error first decreases and then increases. The data suggest that $|u - u_h^*|_{H^1(\Omega)}$ is not $O(h + \eta)$.

We now consider a quadrature rule $Q^i_{nsc}(f)$ for the interval (α_i, β_i) given by

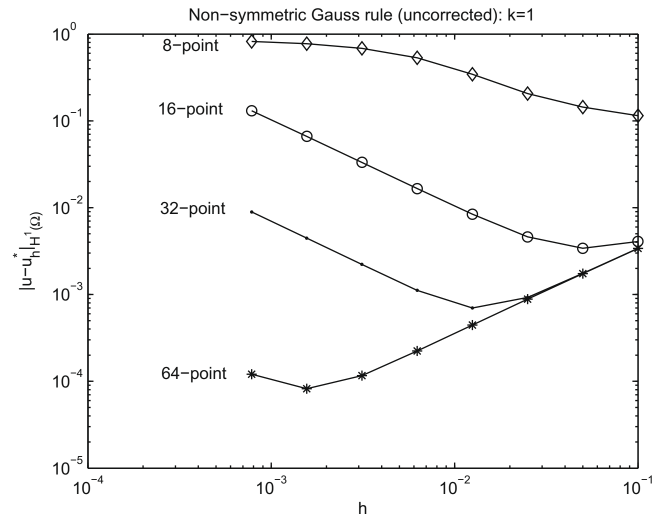

Fig. 2. The loglog plot of $|u - u_h^*|_{H^1(\Omega)}$ with respect to h. u_h^* is the approximate solution obtained using *non-symmetric Gaussian quadrature (uncorrected)* with 8, 16, 32, and 64.

$$Q^i_{nsc}(f) \equiv \sum_{s=1}^{p} \bar{w}_s^{ns} f(\bar{z}_s^{ns})$$

that satisfies (5.4), i.e.,

$$Q^i_{nsc}(\phi'_i) = 0, \quad \forall i \in N''_h. \tag{5.9}$$

Using the ideas presented at the beginning of this section, specifically, using (5.3)–(5.6) and (5.7), we define $\bar{z}_s^{ns} \equiv z_s^{ns}$ and

$$\bar{w}_s^{ns} = w_s^{ns} + \theta_i w_s^{ns} \phi'_i(z_s^{ns}),$$

where z_s^{ns} and w_s^{ns} is defined in (5.8). We choose θ_i, as in (5.7), such that $Q^i_{nsc}(\cdot)$ satisfies (5.9). We refer to $Q^i_{nsc}(f)$ as the ***corrected non-symmetric Gauss-rule***.

We again compute u_h^* – the solution of the meshless method (3.8) with numerical integration, where we used p-point $Q^i_{nsc}(f)$ to compute the relevant integrals. We present the error $|u - u_h^*|_{H^1(\Omega)}$ and the values of h in Table 3. We also present the log–log graph of $|u - u_h^*|_{H^1(\Omega)}$ with respect to h in Fig. 3.

We observe from Table 3 and Fig. 3 that the error $|u - u_h^*|_{H^1(\Omega)}$ behaves differently than the error given in Table 2 and Fig. 2. Moreover, Fig. 3 suggests that $|u - u_h^*|_{H^1(\Omega)} = O(h + \eta)$, which illuminates the main result of this paper for $k = 1$. Thus the data in Tables 2 and 3 strongly suggest that the assumption **QA3** on the numerically quadrature is necessary.

Table 2
Non-Symmetric p-point Gauss rule.

h	$\lvert u - u_h^*\rvert_{H^1(\Omega)}$			
	$p = 8$	$p = 16$	$p = 32$	$p = 64$
1/10	1.1470E−01	4.0789E−03	3.3921E−03	3.3903E−03
1/20	1.4432E−01	3.4066E−03	1.7500E−03	1.7424E−03
1/40	2.0611E−01	4.6066E−03	9.2577E−04	8.8352E−04
1/80	3.4362E−01	8.4159E−03	6.9956E−04	4.4500E−04
1/160	5.3300E−01	1.6622E−02	1.1156E−03	2.2388E−04
1/320	6.8533E−01	3.3220E−02	2.2158E−03	1.1615E−04
1/640	7.7491E−01	6.6244E−02	4.4590E−03	8.2020E−05
1/1280	8.2283E−01	1.3075E−01	8.9503E−03	1.2066E−04

The H^1-seminorm of the error, $|u - u_h^*|_{H^1(\Omega)}$, where $u = e^x - (e-1)$ and u_h^* is the approximate solution obtained using "*non-symmetric Gaussian quadrature*"; the quadrature does not satisfy the assumption (5.4). The shape functions reproduce polynomial of degree $k = 1$.

Table 3
Corrected non-symmetric Gauss rule.

h	$\lvert u - u_h^*\rvert_{H^1(\Omega)}$			
	8 points	16 points	32 points	64 points
1/10	4.8825E−03	3.4363E−03	3.3907E−03	3.3903E−03
1/20	3.4354E−03	1.8473E−03	1.7430E−03	1.7424E−03
1/40	2.7885E−03	1.0985E−03	8.8441E−04	8.8350E−04
1/80	2.5133E−03	8.1143E−04	4.4643E−04	4.4490E−04
1/160	2.3944E−03	7.2840E−04	2.2604E−04	2.2327E−04
1/320	2.3407E−03	7.1009E−04	1.1710E−04	1.1193E−04
1/640	2.3154E−03	7.0752E−04	6.5600E−05	5.6097E−05
1/1280	2.3031E−03	7.0793E−04	4.3936E−05	2.8048E−05

The H^1-seminorm of the error, $|u - u_h^*|_{H^1(\Omega)}$, where $u(x) = e^x - (e-1)$ and u_h^* is the approximate solution obtained using *corrected non-symmetric Gaussian quadrature*. The shape functions reproduce polynomial of degree $k = 1$.

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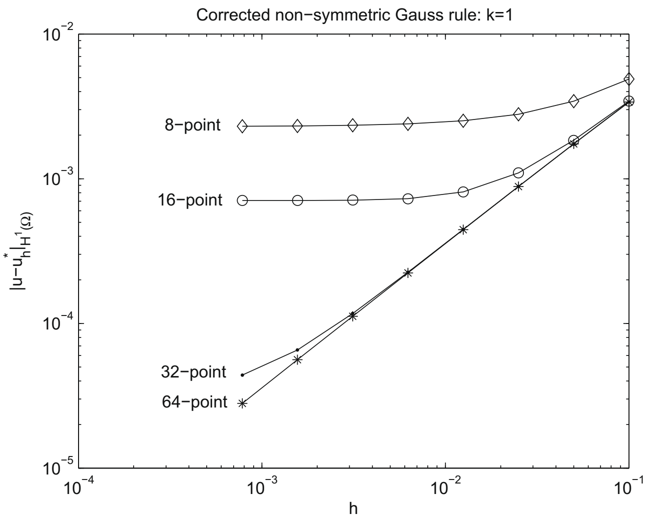

Fig. 3. The loglog plot of $|u-u_h^*|_{H^1(\Omega)}$ with respect to $h.u_h^*$ is the approximate solution obtained using *corrected non-symmetric Gaussian quadrature* with 8, 16, 32, and 64 points.

6. Remarks and conclusions

In this paper, we have developed a mathematical framework to analyze the effect of numerical integration on meshless methods employing shape functions that reproduce polynomials of degree $k \geqslant 1$. The main results are summarized as follows:

- One of our main assumptions on the numerical quadrature is that it satisfy a form of Green's theorem, given in (3.18) in **QA3**. Using numerical integration rules that satisfy (3.18), we have proved error estimates.
- Numerical integration rules, satisfying the assumptions mentioned in this paper, automatically yield the so called "zero row sum condition" (see (3.5)). This was one of the main assumptions that was used to obtain a similar error estimate in [4] for the case $k = 1$.
- Our results indicate that numerical integration with increased accuracy is required as $h \to 0$ to obtain the optimal order of convergence. The numerical results presented in this paper strongly support the results of this paper.

We have considered a scalar second order Neumann boundary value problem with constant coefficient in this paper. The results in this paper can be extended to a general coercive Neumann problem with non-constant coefficients.

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